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Stability estimates for the unique continuation property of the Stokes system and for an inverse boundary coefficient problem

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Abstract
In the first part of this paper, we prove Hölder and logarithmic stability estimates associated with the unique continuation property for the Stokes system. The proof of these results is based on local Carleman inequalities. In the second part, these estimates on the fluid velocity and on the fluid pressure are applied to solve an inverse problem: we consider the Stokes system completed with mixed Neumann and Robin boundary conditions, and we want to recover the Robin coefficient (and obtain the stability estimate for it) from measurements available on a part of the boundary where the Neumann conditions are prescribed. For this identification parameter problem, we obtain a logarithmic stability estimate under the assumption that the velocity of a given reference solution stays far from zero on a part of the boundary where the Robin conditions are prescribed.

1. Introduction

We are interested in stability estimates quantifying unique continuation properties for the Stokes system in a bounded connected open domain $\Omega \subset \mathbb{R}^d$, $d \in \mathbb{N}^*$, as well as their consequences for the stability of a Robin coefficient with respect to measurements available on a part of the boundary. In this work, we will consider the Stokes system:

$$\begin{cases} -\Delta u + \nabla p = 0, & \text{in } \Omega, \\ \operatorname{div} u = 0, & \text{in } \Omega, \end{cases} \tag{1.1}$$

where u and p denote the fluid velocity and the fluid pressure, respectively. For such a system, and more generally for the unsteady Stokes equations with a non-smooth potential, Fabre and Lebeau proved in [18] a unique continuation result. In the particular case of the steady problem (1.1), their result is the following:

Theorem 1.1. *Let ω be a nonempty open set in Ω and $(u, p) \in H^1_{\mathrm{loc}}(\Omega)^d \times L^2_{\mathrm{loc}}(\Omega)$ be a weak solution of system (1.1) satisfying $u = 0$ in ω. Then $u = 0$ and p is constant in Ω.*

We easily deduce from the previous theorem the following result (see [7]).

Corollary 1.2. *Let γ be a nonempty open set included in $\partial\Omega$ and $(u, p) \in H^1(\Omega)^d \times L^2(\Omega)$ be a solution of system (1.1) satisfying $u = 0$ and $\frac{\partial u}{\partial n} - pn = 0$ on γ. Then $u = 0$ and $p = 0$ in Ω.*

One of our purposes is to obtain stability estimates in Ω which quantify these unique continuation results and which are valid for any regular enough solution of (1.1) without extra boundary conditions. More precisely, we obtain two kinds of inequalities. The first inequality stated in the following theorem is a local stability estimate of the Hölder type.

Theorem 1.3. *Let ω be a nonempty open set and K be a compact set, both included in Ω. Then, there exist $c > 0$ and $0 < \beta < 1$, such that for all $(u, p) \in H^1(\Omega)^d \times L^2(\Omega)$ solutions of (1.1), we have*

$$\|u\|_{H^1(K)^d} + \|p\|_{L^2(K)} \leqslant c(\|u\|_{H^1(\omega)^d} + \|p\|_{L^2(\omega)})^{\beta}(\|u\|_{H^1(\Omega)^d} + \|p\|_{L^2(\Omega)})^{1-\beta}. \tag{1.2}$$

Then, we obtain two global logarithmic estimates. In the first one, we estimate the (u, p) solution of (1.1) in the H^1-norm on the whole domain with respect to the L^2-norm of $(u_{|\Gamma}, p_{|\Gamma})$ and $\left(\frac{\partial u}{\partial n}_{|\Gamma}, \frac{\partial p}{\partial n}_{|\Gamma}\right)$, where Γ is a part of the boundary of Ω. In the second one, we obtain an estimate of the (u, p) solution of (1.1) in the H^1-norm on the whole domain with respect to the H^1-norm of u and p in an open set $\omega \subset \Omega$. To be more specific, we prove the following theorem.

Theorem 1.4. *Assume that Ω is of class $\mathcal{C}^\infty$. Let $0 < \nu \leqslant \frac{1}{2}$. Let Γ be a nonempty open subset of the boundary of Ω and ω be a nonempty open set included in Ω. Then, there exists $d_0 > 0$, such that for all $\beta \in \left(0, \frac{1}{2} + \nu\right)$, for all $\tilde{d} > d_0$, there exists $c > 0$, such that we have*

$$\|u\|_{H^1(\Omega)^d} + \|p\|_{H^1(\Omega)} \leqslant c\frac{\|u\|_{H^{\frac{3}{2}+\nu}(\Omega)^d} + \|p\|_{H^{\frac{3}{2}+\nu}(\Omega)}}{\left(\ln\left(\tilde{d}\frac{\|u\|_{H^{\frac{3}{2}+\nu}(\Omega)^d}+\|p\|_{H^{\frac{3}{2}+\nu}(\Omega)}}{\|u\|_{L^2(\Gamma)^d}+\|p\|_{L^2(\Gamma)}+\|\frac{\partial u}{\partial n}\|_{L^2(\Gamma)^d}+\|\frac{\partial p}{\partial n}\|_{L^2(\Gamma)}}\right)\right)^{\beta}} \tag{1.3}$$

and

$$\|u\|_{H^1(\Omega)^d} + \|p\|_{H^1(\Omega)} \leqslant c\frac{\|u\|_{H^{\frac{3}{2}+\nu}(\Omega)^d} + \|p\|_{H^{\frac{3}{2}+\nu}(\Omega)}}{\left(\ln\left(\tilde{d}\frac{\|u\|_{H^{\frac{3}{2}+\nu}(\Omega)^d}+\|p\|_{H^{\frac{3}{2}+\nu}(\Omega)}}{\|u\|_{H^1(\omega)^d}+\|p\|_{H^1(\omega)}}\right)\right)^{\beta}}, \tag{1.4}$$

for all couple $(u, p) \in H^{\frac{3}{2}+\nu}(\Omega)^d \times H^{\frac{3}{2}+\nu}(\Omega)$ solutions of (1.1).

From the point of view of the unique continuation results stated previously, these estimates are not optimal. Indeed, one can note that our stability estimates require more measurements than the Fabre–Lebeau unique continuation result. For instance, in theorem 1.1, the unique continuation result only requires the velocity to be equal to zero, whereas in inequality (1.2), we need information on u and p on ω. Moreover, note that, in (1.3), the constraint $\frac{\partial u}{\partial n} - pn$ which appears in corollary 1.2 is divided into two terms: $\frac{\partial u}{\partial n}$ and pn, and that there is also an additional term, the normal derivative of p. Nevertheless, even if these estimates are not optimal, they are satisfied without prescribing boundary conditions on the solution and have the advantage of providing an upper bound both on u and p. These two points will be crucial in solving the inverse problem of identifying a Robin coefficient defined on some part of the boundary from measurements available on another part of the boundary, where one needs to estimate both u and p. For an optimal three-balls inequality which only involves the L^2-norm of the velocity u, we refer to [24]. Note yet that by applying theorem 1.3 or its quantification obtained in [24], p is only known to be a constant. The result obtained in [24] would not enable us to deal with the inverse problem we are interested in.

As in [25], where quantitative estimates on the unique continuation property for the Laplace equation are established, we use two kinds of local Carleman inequalities to prove theorems 1.3 and 1.4, one near the boundary and one in the interior of the open set Ω. In each case, the method consists in applying the Carleman estimate to u and p simultaneously, by using the fact that $\Delta u = \nabla p$ and $\Delta p = \operatorname{div}(\Delta u) = 0$, in order to free ourselves from terms on the right-hand side of the inequalities. It is interesting to note that if we directly apply the estimate coming from [25] to the (u, p) solution of the Stokes equations, and if we apply the same reasoning as explained above, we obtain ∇p in the L^2-norm over all Ω on the right-hand side of the inequality which we cannot discard. Consequently, we cannot prove theorems 1.3 and 1.4 without going deeply into the heart of the proof. Note that the proof requires the domain to be $\mathcal{C}^\infty$. One could surely consider less regular domains as in [9] where Lipschitz domains (but smooth solutions) are considered for the Laplacian problem. Note furthermore that this $\mathcal{C}^\infty$ regularity is not so restrictive since we will only need the domain to be locally $\mathcal{C}^\infty$ in the parameter identification result (see theorem 1.5).

The second main objective of this paper is to apply the previous stability estimates to some parameter identification problem. We consider the Stokes equations with mixed Neumann and Robin boundary conditions:

$$\begin{cases} -\Delta u + \nabla p = 0, & \text{in } \Omega \\ \operatorname{div} u = 0, & \text{in } \Omega, \\ \dfrac{\partial u}{\partial n} - pn = g, & \text{on } \Gamma_0, \\ \dfrac{\partial u}{\partial n} - pn + qu = 0, & \text{on } \Gamma_{\text{out}}. \end{cases} \tag{1.5}$$

Our aim is to derive stability estimates for the inverse problem of determining the Robin coefficient from the measurements of u and p available on a part of Γ_0. We assume that

$$\Gamma_0 \cup \Gamma_{\text{out}} = \partial\Omega \quad \text{and} \quad \overline{\Gamma}_0 \cap \overline{\Gamma}_{\text{out}} = \emptyset.$$

These assumptions enable us to obtain global regularity on the solution of system (1.5), despite the mixed boundary conditions. As detailed in remark 3.7, under certain conditions, we can relax this assumption.

Let us emphasize that such kinds of systems naturally appear in the modeling of biological problems such as, for example, blood flow in the cardiovascular system (see [26] and [30]) or airflow in the respiratory tract (see [4]). For an introduction to the modeling of the airflow in the lungs and to the different boundary conditions that may be prescribed, we refer to [15]. The fact that no boundary condition is necessary in our previous stability estimates allows us to consider models that are close to applications. The part of the boundary Γ_0 represents a physical boundary on which measurements are available, and Γ_{out} represents an artificial boundary on which Robin boundary conditions (or mixed boundary conditions involving the fluid stress tensor and its flux at the outlet) are prescribed, because no *in vivo* measurements of the velocity u and the pressure p are available. In this case, the Robin coefficient represents in a reduced way the downstream part of the arterial or bronchial tree.

For this problem, we will prove the following logarithmic estimate.

Theorem 1.5. *Let $k \in \mathbb{N}^*$ be such that $k+2 > \frac{d}{2}$, and $s \in \mathbb{R}$ be such that $s > \frac{d-1}{2}$ and $s \geqslant \frac{1}{2}+k$. Let $\Gamma \subseteq \Gamma_0$ be a nonempty open subset of the boundary of Ω. We assume that Γ and Γ_{out} are of class $\mathcal{C}^\infty$. Let $\alpha > 0$, $M_1 > 0$, $M_2 > 0$. We assume that $(g, q_j) \in H^{\frac{1}{2}+k}(\Gamma_0)^d \times H^s(\Gamma_{\text{out}})$,* *for* $j = 1, 2$, *are such that* g *is non-identically zero,* $\|g\|_{H^{\frac{1}{2}+k}(\Gamma_0)^d} \leqslant M_1$, $q_j \geqslant \alpha$ *on* Γ_{out} *and* $\|q_j\|_{H^s(\Gamma_{\mathrm{out}})} \leqslant M_2$. *We denote by* (u_j, p_j) *the solution of system (1.5) with* $q = q_j$, *for* $j = 1, 2$. *Let* K *be a compact subset of* $\{x \in \Gamma_{\mathrm{out}} \mid u_1 \neq 0\}$ *and* $m > 0$ *be such that* $|u_1| \geqslant m$ *on* K.

Then, for all $\beta \in (0, 1)$, *there exist* $C(\alpha, M_1, M_2) > 0$ *and* $C_1(\alpha, M_1, M_2) > 0$, *such that*

$$\|q_1 - q_2\|_{L^2(K)} \leqslant \frac{1}{m} \frac{C(\alpha, M_1, M_2)}{\left(\ln\left(\frac{C_1(\alpha, M_1, M_2)}{\|u_1 - u_2\|_{L^2(\Gamma)^d} + \|p_1 - p_2\|_{L^2(\Gamma)} + \|\frac{\partial p_1}{\partial n} - \frac{\partial p_2}{\partial n}\|_{L^2(\Gamma)}}\right)\right)^{\frac{3}{4}\beta}}. \tag{1.6}$$

Let us note that we obtain an estimate of the Robin parameters on a subset of Γ_{out} where the velocity u_1 stays far from 0. This later assumption can be discarded in very specific cases (see remark 4.9 in [7]) and is generally verified numerically in the considered applications.

Stability estimates for the Robin coefficient have been widely studied for the Laplace equation (see [2, 5, 10–12, 29]). Concerning the Stokes equations, we have obtained in [7] a logarithmic stability estimate valid in dimension 2 for the steady problem as well as the unsteady one, under the assumption that the velocity of a given reference solution stays far from 0 on a part of the boundary where Robin conditions are prescribed. An improvement of this paper is that the stability estimate is valid in any space dimension. Moreover, if we compare the result stated in theorem 1.5 in the particular case $d = 2$ with the previous result in [7], we can note that we need less regularity on the solution (u, p) in theorem 1.5. To be more precise, in [7], the solution (u, p) has to belong to $H^4(\Omega)^d \times H^3(\Omega)$, whereas here, it is sufficient to assume that (u, p) belongs to $H^3(\Omega)^d \times H^2(\Omega)$. Another improvement lies in the fact that the power of the logarithm involved in the stability estimate (1.6) of theorem 1.5 is better than the one obtained in [7]: the power is equal to $3\beta/4$ here, whereas it was equal to $\beta/2$ in [7] for all $\beta \in (0, 1)$.

Let us describe the content of the paper. In section 2, we are concerned with the stability estimates associated with the unique continuation property, and we first state two theorems, namely theorems 2.1 and 2.3, which are equivalent to theorems 1.3 and 1.4, respectively. Then, we state three propositions, namely propositions 2.4, 2.5 and 2.6, which are intermediate results illustrating how information spreads from a part of the boundary to another. These three propositions will allow us to prove theorems 2.1 and 2.3. Proposition 2.4 is based on a local Carleman estimate for the Laplace equation inside the domain, whereas propositions 2.5 and 2.6 are based on a local Carleman estimate near the boundary. The proofs of these propositions are given in subsections 2.1 and 2.2. We conclude the proofs of theorems 2.1 and 2.3 in subsection 2.3. Finally, in section 3, we are concerned with the inverse problem presented above and we give a proof to theorem 1.5.

If not specified otherwise, c is a generic constant, whose value may change and which only depends on the geometry of the open set Ω. Moreover, we denote indifferently by $|\ |$ a norm on $\mathbb{R}^n$ for any $n \geqslant 1$. For $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$, we denote by $x' \in \mathbb{R}^{d-1}$ the $d - 1$ first coordinates of x. We will also use the following notation: $\mathbb{R}^d_+ = \{x = (x', x_d) \in \mathbb{R}^d \mid x_d \geqslant 0\}$.

2. Stability estimates

Let us first state two theorems, theorem 2.1 and theorem 2.3, which are equivalent to theorem 1.3 and theorem 1.4, respectively.

Theorem 2.1. *Let* ω *be a nonempty open set and* K *be a compact set, both included in* Ω. *Then, there exist* $c > 0$ *and* $s > 0$ *such that for all* $(u, p) \in H^1(\Omega)^d \times L^2(\Omega)$ *solutions of (1.1) and for all* $\epsilon > 0$, *we have*

$$\|u\|_{H^1(K)^d} + \|p\|_{L^2(K)} \leqslant \frac{c}{\epsilon}(\|u\|_{H^1(\omega)^d} + \|p\|_{L^2(\omega)}) + \epsilon^s(\|u\|_{H^1(\Omega)^d} + \|p\|_{L^2(\Omega)}). \tag{2.1}$$

This theorem and theorem 1.3 are equivalent. The fact that theorem 2.1 implies theorem 1.3 is a direct consequence of lemma 2.2 below with

$$A = c(\|u\|_{H^1(\omega)^d} + \|p\|_{L^2(\omega)}), \quad B = \|u\|_{H^1(\Omega)^d} + \|p\|_{L^2(\Omega)}, \quad C_1 = 1, \quad C_2 = s,$$
$$\gamma = -\ln\epsilon \quad \text{and} \quad D = \|u\|_{H^1(K)^d} + \|p\|_{L^2(K)}.$$

Moreover, the fact that theorem 1.3 implies theorem 2.1 is a consequence of the Young inequality by writing

$$c(\|u\|_{H^1(\omega)^d} + \|p\|_{L^2(\omega)})^{\beta}(\|u\|_{H^1(\Omega)^d} + \|p\|_{L^2(\Omega)})^{1-\beta}$$
$$= \left(\frac{c}{\epsilon}(\|u\|_{H^1(\omega)^d} + \|p\|_{L^2(\omega)})\right)^{\beta} (\epsilon^{\frac{\beta}{1-\beta}}(\|u\|_{H^1(\Omega)^d} + \|p\|_{L^2(\Omega)}))^{1-\beta}.$$

Lemma 2.2. *Let $A > 0$, $B > 0$, $C_1 > 0$, $C_2 > 0$ and $D > 0$. We assume that there exist $c_0 > 0$ and $\gamma_1 > 0$ such that $D \leqslant c_0 B$, and for all $\gamma \geqslant \gamma_1$,*

$$D \leqslant A\,\mathrm{e}^{C_1\gamma} + B\,\mathrm{e}^{-C_2\gamma}. \tag{2.2}$$

Then, there exists $C > 0$ such that

$$D \leqslant C A^{\frac{C_2}{C_1+C_2}} B^{\frac{C_1}{C_1+C_2}}.$$

Lemma 2.2 will be used repeatedly throughout this paper. We refer to [28] for a proof of this result. Next, we introduce a result which is equivalent to theorem 1.4.

Theorem 2.3. *Assume that Ω is of class $\mathcal{C}^\infty$. Let $0 < \nu \leqslant \frac{1}{2}$, Γ be a nonempty open subset of the boundary of Ω and ω be a nonempty open set included in Ω. Then, for all $\beta \in \left(0, \frac{1}{2} + \nu\right)$, there exists $c > 0$ such that for all $\epsilon > 0$, we have*

$$\|u\|_{H^1(\Omega)^d} + \|p\|_{H^1(\Omega)} \leqslant \mathrm{e}^{\frac{c}{\epsilon}}\left(\|u\|_{L^2(\Gamma)^d} + \|p\|_{L^2(\Gamma)} + \left\|\frac{\partial u}{\partial n}\right\|_{L^2(\Gamma)^d} + \left\|\frac{\partial p}{\partial n}\right\|_{L^2(\Gamma)}\right)$$
$$+ \epsilon^{\beta}(\|u\|_{H^{\frac{3}{2}+\nu}(\Omega)^d} + \|p\|_{H^{\frac{3}{2}+\nu}(\Omega)}) \tag{2.3}$$

and

$$\|u\|_{H^1(\Omega)^d} + \|p\|_{H^1(\Omega)} \leqslant \mathrm{e}^{\frac{c}{\epsilon}}(\|u\|_{H^1(\omega)^d} + \|p\|_{H^1(\omega)}) + \epsilon^{\beta}(\|u\|_{H^{\frac{3}{2}+\nu}(\Omega)^d} + \|p\|_{H^{\frac{3}{2}+\nu}(\Omega)}) \tag{2.4}$$

for all couple $(u, p) \in H^{\frac{3}{2}+\nu}(\Omega)^d \times H^{\frac{3}{2}+\nu}(\Omega)$ solutions of (1.1).

This theorem and theorem 1.4 are equivalent. This relies on classical arguments which can be found in [15] and [25].

We now state three propositions, propositions 2.4, 2.5 and 2.6, which will allow us to prove theorems 2.1 and 2.3. The first proposition allows us to transmit information from an open set to any relatively compact open set in Ω.

Proposition 2.4. *Let ω be a nonempty open set included in Ω and let $\hat{\omega}$ be a relatively compact open set in Ω. Then, there exist $c, s > 0$ such that for all $\epsilon > 0$, for all $(u, p) \in H^1(\Omega)^d \times H^1(\Omega)$ solutions of (1.1),*

$$\|u\|_{H^1(\hat{\omega})^d} + \|p\|_{H^1(\hat{\omega})} \leqslant \frac{c}{\epsilon}(\|u\|_{H^1(\omega)^d} + \|p\|_{H^1(\omega)}) + \epsilon^{s}(\|u\|_{H^1(\Omega)^d} + \|p\|_{H^1(\Omega)}). \tag{2.5}$$

The second proposition allows us to transmit information from a relatively compact open set in Ω to a neighborhood of the boundary.

Proposition 2.5. *Assume that Ω is of class $\mathcal{C}^\infty$. Let $0 < \nu \leqslant \frac{1}{2}$, $x_0 \in \partial\Omega$ and let ω be an open set in Ω. There exists a neighborhood $\hat{\omega}$ of x_0 such that for all $\beta \in \left(0, \frac{1}{2} + \nu\right)$, there exists $c > 0$, such that for all $\epsilon > 0$, for all $(u, p) \in H^{\frac{3}{2}+\nu}(\Omega)^d \times H^{\frac{3}{2}+\nu}(\Omega)$ solutions of (1.1),*

$$\|u\|_{H^1(\hat{\omega}\cap\Omega)^d} + \|p\|_{H^1(\hat{\omega}\cap\Omega)} \leqslant e^{\frac{c}{\epsilon}}(\|u\|_{H^1(\omega)^d} + \|p\|_{H^1(\omega)}) + \epsilon^\beta(\|u\|_{H^{\frac{3}{2}+\nu}(\Omega)^d} + \|p\|_{H^{\frac{3}{2}+\nu}(\Omega)}). \tag{2.6}$$

Finally, the third proposition allows us to transmit information from a part of the boundary of Ω to a relatively compact open set in Ω.

Proposition 2.6. *Assume that Ω is of class $\mathcal{C}^\infty$. Let $0 < \nu \leqslant \frac{1}{2}$, Γ be a nonempty open subset of the boundary of Ω and $\hat{\omega}$ be a relatively compact open set in Ω. Then, there exist $c, s > 0$ such that for all $\epsilon > 0$, for all $(u, p) \in H^{\frac{3}{2}+\nu}(\Omega)^d \times H^{\frac{3}{2}+\nu}(\Omega)$ solutions of (1.1),*

$$\|u\|_{H^1(\hat{\omega})^d} + \|p\|_{H^1(\hat{\omega})} \leqslant \frac{c}{\epsilon}\left(\|u\|_{H^1(\Gamma)^d} + \|p\|_{H^1(\Gamma)} + \left\|\frac{\partial u}{\partial n}\right\|_{L^2(\Gamma)^d} + \left\|\frac{\partial p}{\partial n}\right\|_{L^2(\Gamma)}\right)$$
$$+ \epsilon^s(\|u\|_{H^1(\Omega)^d} + \|p\|_{H^1(\Omega)}).$$

Remark 2.7. The logarithmic nature of inequalities (2.3) and (2.4) comes from proposition 2.5 where an exponential appears in front of the first term of the right-hand side, whereas the estimates in propositions 2.4 and 2.6 lead to Hölder estimates, as a consequence of lemma 2.2.

Theorem 2.3 will be a consequence of propositions 2.4, 2.5 and 2.6, whereas theorem 2.1 will directly come from proposition 2.4 and the use of the Caccioppoli inequality. The next subsection is dedicated to the proof of proposition 2.4. In the second subsection, we prove propositions 2.5 and 2.6. Finally, in the last subsection, we conclude with the proofs of theorems 2.3 and 2.1.

2.1. Estimates on relatively compact open sets: proof of proposition 2.4

Notation 2.8. *Let P be a second-order differential operator defined in an open set M and $\chi \in \mathcal{C}_0^\infty(M)$, such that $\chi = 1$ in a subdomain Π of M. Then, $P(\chi y) = \chi P y + [P, \chi]y$ with $[P, \chi]$ being a first-order operator with support in $M\backslash\Pi$.*

Notation 2.9. *Let $q \in \mathbb{R}^d$, $\delta > 0$ and $0 < \alpha < \alpha'$. We denote by $A_q^\delta(\alpha, \alpha')$ the annulus delimited by the area between two concentric circles of center q and radii $\alpha\delta$ and $\alpha'\delta$, respectively:*

$$A_q^\delta(\alpha, \alpha') = \{x \in \mathbb{R}^d | \alpha\delta < |x - q| < \alpha'\delta\}.$$

Lemma 2.10. *Let $q \in \mathbb{R}^d$, $\delta > 0$ and $(\alpha_i)_{i=1,\dots,5} \in \mathbb{R}^5$ be such that $0 < \alpha_1 < \alpha_2 < \alpha_3 < \alpha_4 < \alpha_5$. Then, there exist $c > 0$, $h_1 > 0$, $c_1 > 0$ and $c_2 > 0$, such that for all $0 < h < h_1$, and for all function $(u, p) \in H^1(B(q, \alpha_5\delta))^d \times H^1(B(q, \alpha_5\delta))$ solutions of*

$$\begin{cases} -\Delta u + \nabla p = 0, & \text{in } B(q, \alpha_5\delta), \\ \operatorname{div} u = 0, & \text{in } B(q, \alpha_5\delta), \end{cases} \tag{2.7}$$

the following inequality is satisfied:

$$\|u\|_{H^1(A_q^\delta(\alpha_2,\alpha_3))^d} + \|p\|_{H^1(A_q^\delta(\alpha_2,\alpha_3))} \leqslant c(e^{c_1/h}(\|u\|_{H^1(B(q,\alpha_2\delta))^d} + \|p\|_{H^1(B(q,\alpha_2\delta))})$$
$$+ e^{-c_2/h}(\|u\|_{H^1(B(q,\alpha_5\delta))^d} + \|p\|_{H^1(B(q,\alpha_5\delta))})), \tag{2.8}$$

with $c_1 = g(\alpha_1\delta) - g(\alpha_3\delta) > 0$ and $c_2 = g(\alpha_3\delta) - g(\alpha_4\delta) > 0$, where $g(x) = e^{-\lambda x^2}$ and λ is large enough.

Proof of lemma 2.10. Let α_0 and α_6 be such that $0 < \alpha_0 < \alpha_1$ and $\alpha_5 < \alpha_6$. We denote by

$$U_0 = A_q^\delta(\alpha_0, \alpha_6), \qquad K_0 = \overline{A_q^\delta(\alpha_1, \alpha_5)}.$$

Let $\chi \in \mathcal{C}_c^\infty(B(q, \alpha_6\delta))$ such that $0 \leqslant \chi \leqslant 1$, $\chi = 1$ on $\Pi = A_q^\delta(\alpha_2, \alpha_4)$ and $\chi = 0$ in the exterior of K_0. We are going to apply the local Carleman estimate inside the domain for the Laplace equation (see [21]) on U_0 and with $\phi(x) = \mathrm{e}^{-\lambda|x-q|^2}$ successively to χu and χp, where (u, p) is the solution of (2.7): there exist $c > 0$ and $h_1 > 0$ such that for all $h \in (0, h_1)$ and for all function $(u, p) \in H^1(B(q, \alpha_5\delta))^d \times H^1(B(q, \alpha_5\delta))$ solutions of (2.7), we have

$$\int_\Pi |u(x)|^2 \,\mathrm{e}^{2\phi(x)/h}\,\mathrm{d}x + h^2 \int_\Pi |\nabla u(x)|^2 \,\mathrm{e}^{2\phi(x)/h}\,\mathrm{d}x \\ \leqslant ch^3 \int_{K_0} |\chi \nabla p(x)|^2 \,\mathrm{e}^{2\phi(x)/h}\,\mathrm{d}x + ch^3 \int_{K_0\setminus\Pi} |[\Delta, \chi]u(x)|^2 \,\mathrm{e}^{2\phi(x)/h}\,\mathrm{d}x, \qquad (2.9)$$

and since $\Delta p = \operatorname{div}(\Delta u) = 0$,

$$\int_\Pi |p(x)|^2 \,\mathrm{e}^{2\phi(x)/h}\,\mathrm{d}x + h^2 \int_{K_0} |\chi \nabla p(x)|^2 \,\mathrm{e}^{2\phi(x)/h}\,\mathrm{d}x \\ \leqslant ch^3 \int_{K_0\setminus\Pi} |[\Delta, \chi]p(x)|^2 \,\mathrm{e}^{2\phi(x)/h}\,\mathrm{d}x + ch^2 \int_{K_0\setminus\Pi} |p(x)|^2 \,\mathrm{e}^{2\phi(x)/h}\,\mathrm{d}x. \qquad (2.10)$$

We add up inequalities (2.9) and (2.10): there exists $h_1 > 0$, such that for all $h \in (0, h_1)$,

$$\mathrm{e}^{g(\alpha_3\delta)/h} \int_{A_q^\delta(\alpha_2,\alpha_3)} (|u(x)|^2 + |p(x)|^2 + h^2(|\nabla u(x)|^2 + |\nabla p(x)|^2)\,\mathrm{d}x \\ \leqslant ch^2\,\mathrm{e}^{g(\alpha_1\delta)/h} \int_{A_q^\delta(\alpha_1,\alpha_2)} |[\Delta, \chi]u(x)|^2 + |[\Delta, \chi]p(x)|^2 + |p(x)|^2\,\mathrm{d}x \\ + ch^2\,\mathrm{e}^{g(\alpha_4\delta)/h} \int_{A_q^\delta(\alpha_4,\alpha_5)} |[\Delta, \chi]u(x)|^2 + |[\Delta, \chi]p(x)|^2 + |p(x)|^2\,\mathrm{d}x.$$

By dividing the previous inequality by h^2, we obtain the desired result. □

Let us introduce the notion of a δ-sequence of balls between two points.

Definition 2.11. *Let $\delta > 0$ and (x_0, x) be two points in Ω. We say that $(B(q_j, \delta))_{j=0,\ldots,N}$ is a δ-sequence of balls between x_0 and x if*

$$\begin{cases} q_0 = x_0, \\ x \in \overline{B(q_N, \delta)}, \\ B(q_{j+1}, \delta) \subset B(q_j, 2\delta) \quad \text{for } j = 0, \ldots, N-1, \\ B(q_j, 3\delta) \subset \Omega. \end{cases}$$

Lemma 2.12. *Let x_0 and x in Ω. There exists $\delta_0 > 0$ such that for all $0 < \delta < \delta_0$, there exists a δ-sequence of balls between x_0 and x.*

Proof of lemma 2.12. We refer to [27] for a proof of this lemma. Let us just mention that in [27], it is asserted that $x \in B(q_N, 2\delta)$, but on looking carefully at the proof, we see that $x \in \overline{B(q_N, \delta)}$. □

We are now able to prove proposition 2.4.

Proof of proposition 2.4. Let $x_0 \in \omega$ and $r_0 > 0$ be such that $B(x_0, r_0) \subset \omega$. For all $x \in \overline{\hat{\omega}}$, there exists, thanks to lemma 2.12, a δ_x-sequence of balls $\left(B\left(q_j^x, \delta_x\right)\right)_{j=0,\ldots,N_x}$ between x_0 and x. Remark that we can assume that $\delta_x < r_0$ for all $x \in \overline{\hat{\omega}}$. The compact $\overline{\hat{\omega}}$ is included in

$\bigcup_{x\in\overline{\hat{\omega}}} B\big(q^x_{N_x},\delta_x\big)$; thus, we can extract a finite subcover: there exist $\kappa \in \mathbb{N}^*$ and $(x_j)_{j=1,\ldots,\kappa} \in \overline{\hat{\omega}}$ such that

$$\overline{\hat{\omega}} \subset \bigcup_{j=1,\ldots,\kappa} B\big(q^j_{N_j},\delta_j\big) \subset \bigcup_{j=1,\ldots,\kappa} B\big(q^j_{N_j},\delta\big), \tag{2.11}$$

where we have denoted $N_j = N_{x_j}, \delta_j = \delta_{x_j}, q^j_i = q^{x_j}_i$ for $j = 1,\ldots,\kappa, i = 0,\ldots,N_j$ and where $\delta = \max_{j=1,\ldots,\kappa} \delta_j$. Remark that we can assume that $N_j = N$ for all $j = 1,\ldots,\kappa$ (if necessary, we consider the same ball several times). Then, by construction, to prove (2.5), it is sufficient to show that there exist $c, s > 0$ such that for all $j = 1,\ldots,\kappa,$ for all $i = 0,\ldots,N-1,$ for all $\epsilon > 0$ and for all $(u, p) \in H^1(\Omega)^d \times H^1(\Omega)$ solutions of (1.1),

$$\|u\|_{H^1(B(q^j_{i+1},\delta))^d} + \|p\|_{H^1(B(q^j_{i+1},\delta))} \leqslant \frac{c}{\epsilon}(\|u\|_{H^1(B(q^j_i,\delta))^d} + \|p\|_{H^1(B(q^j_i,\delta))}) + \epsilon^s(\|u\|_{H^1(\Omega)^d} + \|p\|_{H^1(\Omega)}). \tag{2.12}$$

To prove (2.12), it is sufficient, thanks to the definition of the δ-sequence of balls, to prove that there exist $c, s > 0$ such that for all $j = 1,\ldots,\kappa,$ for all $i = 0,\ldots,N-1,$ for all $\epsilon > 0$ and for all $(u, p) \in H^1(\Omega)^d \times H^1(\Omega)$ solutions of (1.1),

$$\|u\|_{H^1(B(q^j_i,2\delta))^d} + \|p\|_{H^1(B(q^j_i,2\delta))} \leqslant \frac{c}{\epsilon}(\|u\|_{H^1(B(q^j_i,\delta))^d} + \|p\|_{H^1(B(q^j_i,\delta))}) + \epsilon^s\left(\|u\|_{H^1(\Omega)^d} + \|p\|_{H^1(\Omega)}\right). \tag{2.13}$$

Let us emphasize that, thanks to lemma 2.12, we can choose $\delta > 0$ in (2.11) to be small enough, such that $B\big(q^j_i, 5\delta\big) \subset \Omega$ for all $j = 1,\ldots,\kappa$ and $i = 0,\ldots,N-1$ (it is sufficient to take $\delta \leqslant 3\delta_0/5$).

Let $j \in \{1,\ldots,\kappa\}$ and $i \in \{0,\ldots,N\}$. We are going to apply lemma 2.10 with $q = q^j_i$, $\alpha_1 = \frac{1}{4}$, $\alpha_2 = \frac{1}{2}$, $\alpha_3 = 2$, $\alpha_4 = \frac{9}{4}$, $\alpha_5 = \frac{5}{2}$. We find that there exist $c > 0$, $h_1 > 0$, $c_1 = g(\delta/4) - g(2\delta) > 0$ and $c_2 = g(2\delta) - g(9\delta/4) > 0$, such that for all $h \in (0, h_1)$ and for all function $(u, p) \in H^1\big(B\big(q^j_i, 5\delta/2\big)\big)^d \times H^1\big(B\big(q^j_i, 5\delta/2\big)\big)$ solutions of (2.7), we have

$$\|u\|_{H^1(A^\delta_{q^j_i}(\frac{1}{2},2))^d} + \|p\|_{H^1(A^\delta_{q^j_i}(\frac{1}{2},2))} \leqslant c(\mathrm{e}^{c_1/h}(\|u\|_{H^1(B(q^j_i,\delta/2))^d} + \|p\|_{H^1(B(q^j_i,\delta/2))}) + \mathrm{e}^{-c_2/h}(\|u\|_{H^1(B(q^j_i,5\delta/2))^d} + \|p\|_{H^1(B(q^j_i,5\delta/2))})). \tag{2.14}$$

Since $B(q^j_i, 5\delta/2) \subset \Omega$, we obtain

$$\|u\|_{H^1(B(q^j_i,2\delta))^d} + \|p\|_{H^1(B(q^j_i,2\delta))} \leqslant c(\mathrm{e}^{c_1/h}(\|u\|_{H^1(B(q^j_i,\delta))^d} + \|p\|_{H^1(B(q^j_i,\delta))}) + \mathrm{e}^{-c_2/h}(\|u\|_{H^1(\Omega)^d} + \|p\|_{H^1(\Omega)})). \tag{2.15}$$

Let us consider $\epsilon = \mathrm{e}^{-c_1/h}$. We obtain that there exist $c > 0$, $s = \frac{c_2}{c_1} > 0$, such that for all $0 < \epsilon < \epsilon_1 = \mathrm{e}^{-c_1/h_1}$, for all $(u, p) \in H^1(\Omega)^d \times H^1(\Omega)$ solutions of (1.1), we have

$$\|u\|_{H^1(B(q^j_i,2\delta))^d} + \|p\|_{H^1(B(q^j_i,2\delta))} \leqslant c\left(\frac{1}{\epsilon}(\|u\|_{H^1(B(q^j_i,\delta))^d} + \|p\|_{H^1(B(q^j_i,\delta))}) + \epsilon^s(\|u\|_{H^1(\Omega)^d} + \|p\|_{H^1(\Omega)})\right).$$

Since $H^1(\Omega) \hookrightarrow H^1(B(q^j_i, 2\delta))$, this inequality is still valid for $\epsilon \geqslant \epsilon_1$. Thus, we obtain inequality (2.5). □

Remark 2.13. Let $\beta > 0$. If we apply inequality (2.5) with $\epsilon = \epsilon'^{\beta/s}$ and use the fact that $(1/\epsilon')^{\beta/s} \leqslant \mathrm{e}^{c/\epsilon'}$, we note that inequality (2.5) of proposition 2.4 readily implies the same kind of inequality as (2.6) with an exponential weight:

$$\begin{cases} \forall\, \beta > 0, \exists c > 0, \forall\, \epsilon > 0, \quad \forall\,(u, p) \in H^1(\Omega)^d \times H^1(\Omega) \text{ solutions of (1.1)}, \\ \|u\|_{H^1(\hat{\omega})^d} + \|p\|_{H^1(\hat{\omega})} \leqslant \mathrm{e}^{\frac{c}{\epsilon}}(\|u\|_{H^1(\omega)^d} + \|p\|_{H^1(\omega)}) + \epsilon^\beta(\|u\|_{H^1(\Omega)^d} + \|p\|_{H^1(\Omega)}). \end{cases}$$

2.2. Estimates near the boundary: proof of Propositions 2.5 and 2.6

Notation 2.14. *Let* $R_0 \geqslant 0$, $K = \left\{x \in \mathbb{R}^d_+ \,\middle|\, |x| \leqslant R_0\right\}$, $\Sigma = \{x \in \partial K \mid x_d = 0\}$ *and* $S = \partial K\backslash\Sigma$. *We denote by* $H^{\frac{3}{2}+\nu}_{0,S}(K)$ *the restriction to the set K of functions in* $H^{\frac{3}{2}+\nu}_0(B(0, R_0))$.

Lemma 2.15. *Let* $0 < \nu \leqslant \frac{1}{2}$, $0 < r_0 < R_0$, $K = \left\{x \in \mathbb{R}^d_+ \,\middle|\, |x| \leqslant R_0\right\}$, $(f, g) \in L^2(K)^d \times L^2(K)$, $B \in GL_d(\mathcal{C}^\infty(K))$ *and P be a second-order differential operator whose coefficients are* $\mathcal{C}^\infty$ *in a neighborhood of K, defined by* $P(x, \partial_x) = -\partial^2_{x_d} + R(x, \frac{1}{i}\partial_{x'})$. *Let us denote by* $r(x, \xi')$ *the principal symbol of R. We assume that* $r(x, \xi') \in \mathbb{R}$ *and that there exists a constant* $c > 0$ *such that for all* $(x, \xi') \in K \times \mathbb{R}^{d-1}$, *we have* $r(x, \xi') \geqslant c|\xi'|^2$.

We denote by $K(r, r') = \{x \in K \,|\, r < x_d < r'\}$ *for* $0 < r < r' < R_0$. *Then, for all* $\beta \in \left(0, \frac{1}{2} + \nu\right)$, *there exists* $c > 0$ *such that for all* $\epsilon > 0$, *the following inequality holds:*

$$\begin{aligned}\|v\|_{H^1(K(0,r_0))^d} + \|q\|_{H^1(K(0,r_0))} &\\ \leqslant \mathrm{e}^{\frac{c}{\epsilon}}(\|v\|_{H^1(K(r_0,R_0))^d} &+ \|q\|_{H^1(K(r_0,R_0))} + \|f\|_{L^2(K)^d} + \|g\|_{L^2(K)})\\ &+ \epsilon^\beta(\|v\|_{H^{\frac{3}{2}+\nu}(K)^d} + \|q\|_{H^{\frac{3}{2}+\nu}(K)})\end{aligned}$$

for all $(v, q) \in H^{\frac{3}{2}+\nu}_{0,S}(K)^d \times H^{\frac{3}{2}+\nu}_{0,S}(K)$ *solutions of*

$$\begin{cases}-Pv + B\nabla q = f, & \text{in } K,\\ Pq = g, & \text{in } K.\end{cases} \tag{2.16}$$

Proof of lemma 2.15. Let $0 < \epsilon < \epsilon_0 < r_0 < R_0$. We denote by $U = K(0, r_0)$ and $U_\epsilon = K(\epsilon, r_0)$. Let $\chi \in \mathcal{C}^\infty(K)$ be a function equal to zero in K^c, such that $\chi = 1$ in U, $0 \leqslant \chi \leqslant 1$ in $K\backslash U$. We are going to apply successively a local Carleman inequality near the boundary due to Lebeau–Robbiano (see [23]) on K and with $\phi(x) = \mathrm{e}^{\lambda x_d}$ to χv and χq: $\exists\, c > 0, h_1 > 0, \forall\, 0 < h < h_1, \forall\,(v, q) \in H^{\frac{3}{2}+\nu}_{0,S}(K)^d \times H^{\frac{3}{2}+\nu}_{0,S}(K)$ solutions of (2.16),

$$\begin{aligned}\int_U |v(x)|^2\,\mathrm{e}^{2\phi(x)/h}\,\mathrm{d}x &+ h^2\int_U |\nabla v(x)|^2\,\mathrm{e}^{2\phi(x)/h}\,\mathrm{d}x\\ &\leqslant ch^3\int_K |\chi Pv(x)|^2\,\mathrm{e}^{2\phi(x)/h}\,\mathrm{d}x + ch^3\int_{K\backslash U} |[P,\chi]v(x)|^2\,\mathrm{e}^{2\phi(x)/h}\,\mathrm{d}x\\ &\quad + c\int_{\mathbb{R}^{d-1}} (|\chi v(x',0)|^2 + |h\partial_{x'}(\chi v)(x',0)|^2 + |h\partial_{x_d}(\chi v)(x',0)|^2)\,\mathrm{e}^{2\phi(x',0)/h}\,\mathrm{d}x',\end{aligned} \tag{2.17}$$

and

$$\begin{aligned}\int_U |q(x)|^2\,\mathrm{e}^{2\phi(x)/h}\,\mathrm{d}x &+ h^2\int_K |\chi\nabla q(x)|^2\,\mathrm{e}^{2\phi(x)/h}\,\mathrm{d}x\\ &\leqslant ch^3\int_K |\chi Pq(x)|^2\,\mathrm{e}^{2\phi(x)/h}\,\mathrm{d}x + ch^3\int_{K\backslash U} |[P,\chi]q(x)|^2\,\mathrm{e}^{2\phi(x)/h}\,\mathrm{d}x\\ &\quad + ch^2\int_{K\backslash U} |q(x)|^2\,\mathrm{e}^{2\phi(x)/h}\,\mathrm{d}x\\ &\quad + c\int_{\mathbb{R}^{d-1}} (|\chi q(x',0)|^2 + |h\partial_{x'}(\chi q)(x',0)|^2 + |h\partial_{x_d}(\chi q)(x',0)|^2)\,\mathrm{e}^{2\phi(x',0)/h}\,\mathrm{d}x'.\end{aligned} \tag{2.18}$$

By summing up inequalities (2.17) and (2.18), dividing by h^2, replacing $\phi(x)$ by $\mathrm{e}^{\lambda x_d}$ and thanks to the trace inequality, we obtain, for h being small enough,

$$
\begin{aligned}
&e^{\frac{e^{\lambda\epsilon}}{h}}(\|v\|_{H^1(K(\epsilon,r_0))^d}+\|q\|_{H^1(K(\epsilon,r_0))})\\
&\qquad\leqslant c\,e^{\frac{e^{\lambda R_0}}{h}}(\|v\|_{H^1(K(r_0,R_0))^d}+\|q\|_{H^1(K(r_0,R_0))}+\|f\|_{L^2(K)^d}+\|g\|_{L^2(K)})\\
&\qquad\quad+\frac{c}{h}\,e^{\frac{1}{h}}(\|v\|_{H^{\frac{3}{2}+\nu}(K)^d}+\|q\|_{H^{\frac{3}{2}+\nu}(K)}).
\end{aligned}
$$

Remark that for all $\epsilon \geqslant 0$, $-e^{\lambda\epsilon}+1 \leqslant -\epsilon$ as long as λ is large enough. Thus,

$$
\begin{aligned}
&\|v\|_{H^1(K(\epsilon,r_0))^d}+\|q\|_{H^1(K(\epsilon,r_0))}\\
&\qquad\leqslant c\,e^{\frac{c}{h}}(\|v\|_{H^1(K(r_0,R_0))^d}+\|q\|_{H^1(K(r_0,R_0))}+\|f\|_{L^2(K)^d}+\|g\|_{L^2(K)})\\
&\qquad\quad+\frac{c}{h}\,e^{-\frac{\epsilon}{h}}(\|v\|_{H^{\frac{3}{2}+\nu}(K)^d}+\|q\|_{H^{\frac{3}{2}+\nu}(K)}).
\end{aligned}
$$

Moreover, for all $\epsilon \geqslant 0$, $\frac{1}{h} \leqslant \frac{2}{\epsilon}e^{\frac{\epsilon}{2h}}$, which implies

$$
\begin{aligned}
&\|v\|_{H^1(K(\epsilon,r_0))^d}+\|q\|_{H^1(K(\epsilon,r_0))}\\
&\qquad\leqslant c\,e^{\frac{c}{h}}(\|v\|_{H^1(K(r_0,R_0))^d}+\|q\|_{H^1(K(r_0,R_0))}+\|f\|_{L^2(K)^d}+\|g\|_{L^2(K)})\\
&\qquad\quad+\frac{c}{\epsilon}\,e^{-\frac{\epsilon}{2h}}(\|v\|_{H^{\frac{3}{2}+\nu}(K)^d}+\|q\|_{H^{\frac{3}{2}+\nu}(K)}).
\end{aligned}
$$

According to lemma 2.2, we obtain

$$
\begin{aligned}
&\|v\|_{H^1(K(\epsilon,r_0))^d}+\|q\|_{H^1(K(\epsilon,r_0))}\\
&\qquad\leqslant c(\|v\|_{H^1(K(r_0,R_0))^d}+\|q\|_{H^1(K(r_0,R_0))}+\|f\|_{L^2(K)^d}+\|g\|_{L^2(K)})^{\frac{\epsilon}{\epsilon+c}}\\
&\qquad\quad\times\left(\frac{1}{\epsilon}(\|v\|_{H^{\frac{3}{2}+\nu}(K)^d}+\|q\|_{H^{\frac{3}{2}+\nu}(K)})\right)^{1-\frac{\epsilon}{\epsilon+c}}.
\end{aligned}
$$

Let $s > 0$ and $\mu > 1$. The previous estimate can be rewritten as

$$
\begin{aligned}
&\|v\|_{H^1(K(\epsilon,r_0))^d}+\|q\|_{H^1(K(\epsilon,r_0))}\\
&\qquad\leqslant c(\epsilon^{-\frac{c}{\epsilon}(s+1)}(\|v\|_{H^1(K(r_0,R_0))^d}+\|q\|_{H^1(K(r_0,R_0))}+\|f\|_{L^2(K)^d}+\|g\|_{L^2(K)}))^{\frac{\epsilon}{\epsilon+c}}\\
&\qquad\quad\times(\epsilon^{s}(\|v\|_{H^{\frac{3}{2}+\nu}(K)^d}+\|q\|_{H^{\frac{3}{2}+\nu}(K)}))^{1-\frac{\epsilon}{\epsilon+c}}\\
&\qquad\leqslant c\epsilon^{-\frac{c(s+1)}{\epsilon}}(\|v\|_{H^1(K(r_0,R_0))^d}+\|q\|_{H^1(K(r_0,R_0))}+\|f\|_{L^2(K)^d}+\|g\|_{L^2(K)})\\
&\qquad\quad+\epsilon^{s}(\|v\|_{H^{\frac{3}{2}+\nu}(K)^d}+\|q\|_{H^{\frac{3}{2}+\nu}(K)}).
\end{aligned}
$$

But $\epsilon^{-\frac{c(s+1)}{\epsilon}} = \exp\left(\frac{c}{\epsilon}(s+1)\ln\left(\frac{1}{\epsilon}\right)\right) \leqslant \exp\left(\frac{c(s+1)}{(\mu-1)\epsilon^{\mu}}\right)$ for ϵ being small enough. Finally, for all $s > 0$, for all $\mu > 1$, there exists $c > 0$, such that for all $0 < \epsilon < \epsilon_0$,

$$
\begin{aligned}
&\|v\|_{H^1(K(\epsilon,r_0))^d}+\|q\|_{H^1(K(\epsilon,r_0))}\\
&\qquad\leqslant c\,e^{\frac{c}{\epsilon^{\mu}}}(\|v\|_{H^1(K(r_0,R_0))^d}+\|q\|_{H^1(K(r_0,R_0))}+\|f\|_{L^2(K)^d}+\|g\|_{L^2(K)})\\
&\qquad\quad+\epsilon^{s}(\|v\|_{H^{\frac{3}{2}+\nu}(K)^d}+\|q\|_{H^{\frac{3}{2}+\nu}(K)})
\end{aligned}
\tag{2.19}
$$

for all (v, q) solutions of (2.16). It remains to estimate $\|v\|_{H^1(K(0,\epsilon))}+\|q\|_{H^1(K(0,\epsilon))}$ uniformly in ϵ. This is a consequence of the Hardy inequality (see [13]) that we recall below.

Lemma 2.16 (Hardy inequality)**.** *Let $0 < \tau < \frac{1}{2}$. There exists $c > 0$ such that for all $h \in H^{\tau}(\mathbb{R}^d_+)$, we have*

$$
\left\|\frac{h}{x_d^{\tau}}\right\|_{L^2(\mathbb{R}^d_+)} \leqslant c\|h\|_{H^{\tau}(\mathbb{R}^d_+)}.
$$

We extend v and q by zero in $\mathbb{R}^d_+\backslash K$. Note that these extensions, denoted respectively by $\tilde{v}$ and $\tilde{q}$, belong to $H^{\frac{3}{2}+\nu}(\mathbb{R}^d_+)$ (see [17]). Let $\tilde{\chi}$ be a function which belongs to $\mathcal{C}^{\infty}_c(\{(x', x_d) \in \mathbb{R}^d_+/x_d < r_0\})$, such that $\tilde{\chi} = 1$ on $K(0, \epsilon)$ and $0 \leqslant \tilde{\chi} \leqslant 1$ elsewhere.

The functions $\tilde{\chi}\tilde{v}$ and $\tilde{\chi}\tilde{q}$ belong to $H^{\frac{3}{2}+\nu}(\mathbb{R}^d_+)$; therefore, as a result of the Hardy inequality, we have that for all $0 < \tau < \frac{1}{2}$, there exists $c > 0$ such that

$$\left\| \frac{v}{x_d^\tau} \right\|_{L^2(K(0,\epsilon))^d} \leqslant \left\| \frac{\tilde{\chi}\tilde{v}}{x_d^\tau} \right\|_{L^2(\mathbb{R}^d_+)^d} \leqslant c\|\tilde{\chi}\tilde{v}\|_{H^\tau(\mathbb{R}^d_+)^d}.$$

Since $\tilde{\chi}\tilde{v} = 0$ in $\left(\mathbb{R}^d_+\backslash K\right) \cup K(r_0, R_0)$, we obtain

$$\left\| \frac{v}{x_d^\tau} \right\|_{L^2(K(0,\epsilon))^d} \leqslant c\|v\|_{H^\tau(K(0,r_0))^d} \leqslant c\|v\|_{H^{\frac{1}{2}}(K(0,r_0))^d}.$$

Consequently, for all $\tau \in \left(0, \frac{1}{2}\right)$, there exists $c > 0$ such that for all $\alpha > 0$,

$$\begin{aligned}\|v\|_{L^2(K(0,\epsilon))^d} &\leqslant c\epsilon^\tau\|v\|_{H^{\frac{1}{2}}(K(0,r_0))^d} \leqslant c\epsilon^\tau\|v\|^{\frac{1}{2}}_{H^1(K)^d}\|v\|^{\frac{1}{2}}_{L^2(K(0,r_0))^d}\\ &\leqslant c\left(\frac{\epsilon^{2\tau}}{\alpha}\|v\|_{H^1(K)^d} + \alpha\|v\|_{L^2(K(0,r_0))^d}\right),\end{aligned}$$

where we used an interpolation inequality and the Young inequality. In the same way, we have for ∇v

$$\begin{aligned}\|\nabla v\|_{L^2(K(0,\epsilon))^{d\times d}} &\leqslant c\epsilon^\tau\|\nabla v\|_{H^{\frac{1}{2}}(K(0,r_0))^{d\times d}}\\ &\leqslant c\left(\epsilon^{\tau(1+2\nu)}\frac{1}{\alpha^{2\nu}}\|v\|_{H^{\frac{3}{2}+\nu}(K)^d} + \alpha\|v\|_{H^1(K(0,r_0))^d}\right).\end{aligned}$$

To summarize, for all $\beta \in \left(0, \frac{1}{2}+\nu\right)$, there exists $c > 0$ such that for all $0 < \alpha < 1$,

$$\|v\|_{H^1(K(0,\epsilon))^d} \leqslant c\left(\frac{\epsilon^\beta}{\alpha}\|v\|_{H^{\frac{3}{2}+\nu}(K)^d} + \alpha\|v\|_{H^1(K(0,r_0))^d}\right).$$

The same inequality also holds for q. Thus, for all $\beta \in \left(0, \frac{1}{2}+\nu\right)$, there exists $c > 0$, such that for all $0 < \alpha < 1$,

$$\begin{aligned}&\|v\|_{H^1(K(0,\epsilon))^d} + \|q\|_{H^1(K(0,\epsilon))}\\ &\quad\leqslant c\left(\frac{\epsilon^\beta}{\alpha}(\|v\|_{H^{\frac{3}{2}+\nu}(K)^d} + \|q\|_{H^{\frac{3}{2}+\nu}(K)}) + \alpha(\|v\|_{H^1(K(0,r_0))^d} + \|q\|_{H^1(K(0,r_0))})\right).\end{aligned}\tag{2.20}$$

We can choose α to be small enough such that by combining (2.19) and (2.20) we have that for all $\beta \in \left(0, \frac{1}{2}+\nu\right)$, for all $\mu > 1$, there exists $c > 0$, such that for all $0 < \epsilon < \epsilon_0$,

$$\begin{aligned}&\|v\|_{H^1(K(0,r_0))^d} + \|q\|_{H^1(K(0,r_0))}\\ &\quad\leqslant c\,\mathrm{e}^{\frac{c}{\epsilon^\mu}}(\|v\|_{H^1(K(r_0,R_0))^d} + \|q\|_{H^1(K(r_0,R_0))} + \|f\|_{L^2(K)^d} + \|g\|_{L^2(K)})\\ &\quad\quad + c\epsilon^\beta(\|v\|_{H^{\frac{3}{2}+\nu}(K)^d} + \|q\|_{H^{\frac{3}{2}+\nu}(K)}).\end{aligned}$$

By a change of variables, we obtain that, for all $\beta \in \left(0, \frac{1}{2}+\nu\right)$, there exists $c > 0$, such that for all $0 < \epsilon < \tilde{\epsilon}_0$,

$$\begin{aligned}&\|v\|_{H^1(K(0,r_0))^d} + \|q\|_{H^1(K(0,r_0))}\\ &\quad\leqslant \mathrm{e}^{\frac{c}{\epsilon}}(\|v\|_{H^1(K(r_0,R_0))^d} + \|q\|_{H^1(K(r_0,R_0))} + \|f\|_{L^2(K)^d} + \|g\|_{L^2(K)})\\ &\quad\quad + \epsilon^\beta(\|v\|_{H^{\frac{3}{2}+\nu}(K)^d} + \|q\|_{H^{\frac{3}{2}+\nu}(K)}).\end{aligned}\tag{2.21}$$

At last, we note that, since $H^{\frac{3}{2}+\nu}(K) \hookrightarrow H^1(K(0, r_0))$, this last inequality remains true for $\epsilon \geqslant \tilde{\epsilon}_0$. □

Let us now prove proposition 2.5.

Proof of proposition 2.5. We are first going to prove that there exist an open neighborhood $\hat{\omega}$ of x_0 and two relatively compact open sets $\tilde{\omega}_1 \subset \Omega$ and $\tilde{\omega}_2 \subset \Omega$, such that for all $\beta \in \left(0, \frac{1}{2}+\nu\right)$, there exists $c > 0$, such that for all $\epsilon > 0$ and for all $(u, p) \in H^{\frac{3}{2}+\nu}(\Omega)^d \times H^{\frac{3}{2}+\nu}(\Omega)$ solutions of (1.1),

$$\begin{aligned}\|u\|_{H^1(\hat{\omega}\cap\Omega)^d} + \|p\|_{H^1(\hat{\omega}\cap\Omega)} \leqslant e^{\frac{c}{\epsilon}}(\|u\|_{H^1(\tilde{\omega}_1)^d} + \|p\|_{H^1(\tilde{\omega}_1)} + \|u\|_{H^1(\tilde{\omega}_2)^d} + \|p\|_{H^1(\tilde{\omega}_2)})\\ + \epsilon^\beta(\|u\|_{H^{\frac{3}{2}+\nu}(\Omega)^d} + \|p\|_{H^{\frac{3}{2}+\nu}(\Omega)}).\end{aligned} \tag{2.22}$$

Then, to pass from (2.22) to (2.6) to obtain the estimate for any ω, it is sufficient to apply inequality (2.5) of proposition 2.4.

Let $\mathcal{V}$ be a neighborhood of x_0 such that $\Omega \cap \mathcal{V} = \{(x', x_d) \in \mathcal{V} \mid x_d > \sigma(x')\}$ with $\sigma \in \mathcal{C}^\infty$. By using the normal geodesic coordinates, it is possible to straighten locally in a neighborhood $\mathcal{V}$ of x_0 the Laplace operator and the boundary simultaneously. Restricting, if necessary, the open set $\mathcal{V}$, we can assume that there exists a neighborhood $\tilde{\mathcal{V}} \subset \mathcal{V}$ of x_0, a surface S such that $S\cap\tilde{\mathcal{V}} = \partial\Omega\cap\tilde{\mathcal{V}}$, and S is deformed inwardly in the open set Ω in $\mathcal{V}\backslash\tilde{\mathcal{V}}$ (this means that there exists $s \in \mathcal{C}^\infty$ such that $S = \{(x', x_d) \in \mathcal{V} \mid x_d = s(x')\}$ with $s = \sigma$ in $\tilde{\mathcal{V}}$ and $s > \sigma$ in $\mathcal{V}\backslash\tilde{\mathcal{V}}$) and a diffeomorphism, denoted ψ, which straightens both S and the Laplace operator. Let us denote by $\tilde{\Omega} = \{(x', x_d) \in \mathcal{V} \mid x_d > s(x')\}$. Note that, by construction, there exists $0 < r_3 < R_0$, such that $\psi^{-1}(\{x \in K \mid r_3 < |x|\})$ is a relatively compact open set of Ω and $K = \{x \in \mathbb{R}^d_+ | \, |x| \leqslant R_0\} \subset \psi(\tilde{\Omega})$. Let $\xi \in \mathcal{C}^\infty_c(\overline{K})$ be such that $\xi = 1$ in $\{x \in \mathbb{R}^d_+ \mid |x| \leqslant r_3\}$ and $0 \leqslant \xi \leqslant 1$ elsewhere. Let us denote by $\varrho = \xi \circ \psi$. Note that since $(v, q) = (\varrho u, \varrho p)$ is the solution in $\tilde{\Omega} \cap \mathcal{V}$ of

$$\begin{cases}-\Delta v + \nabla q = f,\\ \Delta q = g,\end{cases}$$

with $f = -u\Delta\varrho - 2\nabla u\nabla\varrho + \nabla\varrho p$ and $g = \Delta\varrho p + 2\nabla\varrho\cdot\nabla p$, then $(w, \pi) = ((\varrho u)\circ\psi^{-1}, (\varrho p)\circ\psi^{-1})$ is the solution in K of

$$\begin{cases}-Pw + (\nabla\psi)^T\nabla\pi = f\circ\psi^{-1},\\ P\pi = g\circ\psi^{-1}.\end{cases} \tag{2.23}$$

We apply lemma 2.15 to (w, π). We obtain that for all $\beta \in \left(0, \frac{1}{2}+\nu\right)$, there exists $c > 0$, such that for all $\epsilon > 0$,

$$\begin{aligned}&\|w\|_{H^1(K(0,r_0)\cap B(0,r_3))^d} + \|\pi\|_{H^1(K(0,r_0)\cap B(0,r_3))}\\ &\quad\leqslant e^{\frac{c}{\epsilon}}(\|w\|_{H^1(K(r_0,R_0))^d} + \|\pi\|_{H^1(K(r_0,R_0))} + \|f\circ\psi^{-1}\|_{L^2(K)^d} + \|g\circ\psi^{-1}\|_{L^2(K)})\\ &\quad\quad + \epsilon^\beta(\|w\|_{H^{\frac{3}{2}+\nu}(K)^d} + \|\pi\|_{H^{\frac{3}{2}+\nu}(K)}).\end{aligned}$$

In other words, there exist an open neighborhood $\hat{\omega}$ of x_0 and a relatively compact open set $\tilde{\omega}_1 \subset \Omega$, such that for all $\beta \in \left(0, \frac{1}{2}+\nu\right)$, there exists $c > 0$, such that for all $\epsilon > 0$ and for all $(u, p) \in H^{\frac{3}{2}+\nu}(\Omega)^d \times H^{\frac{3}{2}+\nu}(\Omega)$ solutions of (1.1),

$$\begin{aligned}\|u\|_{H^1(\hat{\omega}\cap\Omega)^d} + \|p\|_{H^1(\hat{\omega}\cap\Omega)} \leqslant e^{\frac{c}{\epsilon}}(\|u\|_{H^1(\tilde{\omega}_1)^d} + \|p\|_{H^1(\tilde{\omega}_1)} + \|f\|_{L^2(\tilde{\Omega})^d} + \|g\|_{L^2(\tilde{\Omega})})\\ + \epsilon^\beta(\|u\|_{H^{\frac{3}{2}+\nu}(\Omega)^d} + \|p\|_{H^{\frac{3}{2}+\nu}(\Omega)}).\end{aligned}$$

To conclude, let us remark that since $\xi = 1$ in $\left\{x \in \mathbb{R}^d_+ \,\middle|\, |x| \leqslant r_3\right\}$, $\mathrm{supp}(\nabla\xi) \subset \{x \in K \mid r_3 < |x|\}$ and then $\mathrm{supp}(\nabla\varrho) \subset \psi^{-1}(\{x \in K \mid r_3 < |x|\})$, which is a relatively compact open set of Ω. Then, recalling the definition of f and g, we obtain that there exists a relatively compact open set $\tilde{\omega}_2 \subset \Omega$ such that for all $\beta \in \left(0, \frac{1}{2} + \nu\right)$, there exists $c > 0$, such that for all $\epsilon > 0$ and for all $(u, p) \in H^{\frac{3}{2}+\nu}(\Omega)^d \times H^{\frac{3}{2}+\nu}(\Omega)$ solutions of (1.1),

$$\|u\|_{H^1(\hat{\omega}\cap\Omega)^d} + \|p\|_{H^1(\hat{\omega}\cap\Omega)} \leqslant e^{\frac{c}{\epsilon}}(\|u\|_{H^1(\tilde{\omega}_1)^d} + \|p\|_{H^1(\tilde{\omega}_1)} + \|u\|_{H^1(\tilde{\omega}_2)^d} + \|p\|_{H^1(\tilde{\omega}_2)})$$
$$+ \epsilon^\beta(\|u\|_{H^{\frac{3}{2}+\nu}(\Omega)^d} + \|p\|_{H^{\frac{3}{2}+\nu}(\Omega)}).$$

□

Let us end this subsection with the proof of proposition 2.6.

Proof of proposition 2.6. Let $x_0 \in \Gamma$. We are going to prove that there exists a neighborhood ω of x_0 such that there exist $c, s > 0$ such that for all $\epsilon > 0$, for all $(u, p) \in H^{\frac{3}{2}+\nu}(\Omega)^d \times H^{\frac{3}{2}+\nu}(\Omega)$ solutions of (1.1),

$$\|u\|_{H^1(\omega\cap\Omega)^d} + \|p\|_{H^1(\omega\cap\Omega)} \leqslant \frac{c}{\epsilon}\left(\|u\|_{H^1(\Gamma)^d} + \|p\|_{H^1(\Gamma)} + \left\|\frac{\partial u}{\partial n}\right\|_{L^2(\Gamma)^d} + \left\|\frac{\partial p}{\partial n}\right\|_{L^2(\Gamma)}\right)$$
$$+ \epsilon^s(\|u\|_{H^1(\Omega)^d} + \|p\|_{H^1(\Omega)}), \tag{2.24}$$

which implies proposition 2.6 thanks to inequality (2.5) of proposition 2.4.

Near the boundary, in a neighborhood of x_0, we go back to the half-plane, thanks to geodesic normal coordinates: let ψ, $R_0 > 0$ and $\mathcal{V}$ be such that $\psi(\Omega \cap \mathcal{V}) = \left\{x \in \mathbb{R}^d_+ \,\middle|\, |x| < R_0\right\} = \mathring{K}$ and $\psi(\partial\Omega\cap\mathcal{V}) = \{(x', x_d) \in \mathbb{R}^d \mid x_d = 0 \text{ and } |x| < R_0\}$. We can always assume that $\mathcal{V}$ is small enough to have $\partial\Omega\cap\mathcal{V} \subset \Gamma$. In the following, we denote by $\Sigma = \psi(\partial\Omega\cap\mathcal{V}) \subset \mathbb{R}^{d-1}$ and by $(v, q) = (u \circ \psi^{-1}, p \circ \psi^{-1})$. Note that (v, q) is the solution in K of

$$\begin{cases} -Pv + (\nabla\psi)^{\mathrm{T}}\nabla q = 0, \\ Pq = 0. \end{cases} \tag{2.25}$$

We are going to prove that there exists a neighborhood θ of 0 such that for all $\epsilon > 0$, for all $(v, q) \in H^{\frac{3}{2}+\nu}(K)^d \times H^{\frac{3}{2}+\nu}(K)$ solutions of (2.25),

$$\|v\|_{H^1(K\cap\theta)^d} + \|q\|_{H^1(K\cap\theta)} \leqslant \frac{c}{\epsilon}(\|v\|_{H^1(\Sigma)^d} + \|q\|_{H^1(\Sigma)} + \|\partial_{x_d}v\|_{L^2(\Sigma)^d} + \|\partial_{x_d}q\|_{L^2(\Sigma)})$$
$$+ \epsilon^s(\|v\|_{H^1(K)^d} + \|q\|_{H^1(K)}).$$

Let $U = \{x \in K \mid x_d + |x|^2 \leqslant r_0\}$ with r_0 being small enough and $\chi \in \mathcal{C}_c^\infty(\overline{K})$ be such that $\chi = 1$ on U, $0 \leqslant \chi \leqslant 1$ in $K\backslash U$. By the successive application of a local Carleman inequality due to Lebeau–Robbiano (see [27]) on K and with $\phi = e^{-\lambda(x_d+|x|^2)}$ to χv and to χq, we obtain (in the same way as in the proof of lemma 2.15) that there exist $c > 0$, $h_1 > 0$, such that for all $0 < h < h_1$, for all $(v, q) \in H^{\frac{3}{2}+\nu}(K)^d \times H^{\frac{3}{2}+\nu}(K)$ satisfying (2.25),

$$\int_U (|v(x)|^2 + |q(x)|^2)\, e^{2\phi(x)/h}\, dx + h^2 \int_U (|\nabla v(x)|^2 + |\nabla q(x)|^2)\, e^{2\phi(x)/h}\, dx$$
$$\leqslant ch^3 \int_{K\backslash U} |\nabla q(x)|^2\, e^{2\phi(x)/h}\, dx + ch^2 \int_{K\backslash U} |q(x)|^2\, e^{2\phi(x)/h}\, dx$$
$$+ ch^3 \int_{K\backslash U} (|[P, \chi]v(x)|^2 + |[P, \chi]q(x)|^2)\, e^{2\phi(x)/h}\, dx$$
$$+ c\int_{\mathbb{R}^{d-1}} (|h\partial_{x'}(\chi v)(x', 0)|^2 + |h\partial_{x'}(\chi q)(x', 0)|^2 + |h\partial_{x_d}(\chi v)(x', 0)|^2$$
$$+ |h\partial_{x_d}(\chi q)(x', 0)|^2)\, e^{2\phi(x',0)/h}\, dx'$$
$$+ c\int_{\mathbb{R}^{d-1}} (|\chi v(x', 0)|^2 + |\chi q(x', 0)|^2)\, e^{2\phi(x',0)/h}\, dx'.$$

We denote by $R(r, r') = \{x \in K \mid r < x_d + |x|^2 < r'\}$. The previous inequality becomes, with $0 < z_1 < r_0 < z_2 < R_0$,

$$\mathrm{e}^{\frac{\mathrm{e}^{-\lambda z_1}}{h}}(\|v\|_{H^1(R(0,z_1))^d} + \|q\|_{H^1(R(0,z_1))}) \leqslant c\,\mathrm{e}^{\frac{\mathrm{e}^{-\lambda z_2}}{h}}(\|v\|_{H^1(R(z_2,R_0))^d} + \|q\|_{H^1(R(z_2,R_0))})$$
$$+ c\,\mathrm{e}^{\frac{1}{h}}(\|v\|_{H^1(\Sigma)^d} + \|\partial_{x_d} v\|_{L^2(\Sigma)^d} + \|q\|_{H^1(\Sigma)} + \|\partial_{x_d} q\|_{L^2(\Sigma)}).$$

Accordingly, there exist $c, h_1 > 0$, such that for all $0 < h < h_1$, for all $(v, q) \in H^{\frac{3}{2}+\nu}(\Omega)^d \times H^{\frac{3}{2}+\nu}(\Omega)$ solutions of (2.25),

$$\|v\|_{H^1(R(0,z_1))^d} + \|q\|_{H^1(R(0,z_1))} \leqslant c\,\mathrm{e}^{-\frac{1}{h}}(\|v\|_{H^1(K)^d} + \|q\|_{H^1(K)})$$
$$+ c\,\mathrm{e}^{\frac{c}{h}}(\|v\|_{H^1(\Sigma)^d} + \|\partial_{x_d} v\|_{L^2(\Sigma)^d} + \|q\|_{H^1(\Sigma)} + \|\partial_{x_d} q\|_{L^2(\Sigma)}).$$

We can conclude the proof in the same way as we concluded the proof of inequality (2.5): we obtain inequality (2.24) with $\omega \cap \Omega = \psi^{-1}(R(0, z_1))$. □

2.3. Global estimates

In this subsection, we conclude the proofs of theorems 2.1 and 2.3. Let us first prove theorem 2.3.

Proof of theorem 2.3. Let $\hat{\omega}$ be a relatively compact open set in Ω. For each $x \in \partial\Omega$, we deduce from proposition 2.5 that there exists a neighborhood ω_x of x, such that for all $\beta \in \left(0, \frac{1}{2} + \nu\right)$, there exists $c > 0$, such that for all $\epsilon > 0$ and for all $(u, p) \in H^{\frac{3}{2}+\nu}(\Omega)^d \times H^{\frac{3}{2}+\nu}(\Omega)$ solutions of (1.1), inequality (2.6) is satisfied. We point out that $\partial\Omega \subseteq \bigcup_{x\in\partial\Omega} w_x$ and that $\partial\Omega$ is compact. Thus, we can extract a finite subcover: there exist $N \in \mathbb{N}$ and $x_i \in \partial\Omega$, $i = 1, \ldots, N$, such that $\partial\Omega \subset \bigcup_{i=1}^N \omega_{x_i}$. For $i = 1, \ldots, N$, let us denote by $\omega_i = \omega_{x_i}$. As a result, we obtain that for all $\beta \in \left(0, \frac{1}{2} + \nu\right)$, there exists $c > 0$, such that for all $i \in \{1, \ldots, N\}$, for all $\epsilon > 0$, for all $(u, p) \in H^{\frac{3}{2}+\nu}(\Omega)^d \times H^{\frac{3}{2}+\nu}(\Omega)$ solutions of (1.1),

$$\|u\|_{H^1(\omega_i\cap\Omega)^d} + \|p\|_{H^1(\omega_i\cap\Omega)} \leqslant \mathrm{e}^{\frac{c}{\epsilon}}(\|u\|_{H^1(\hat{\omega})^d} + \|p\|_{H^1(\hat{\omega})}) + \epsilon^\beta(\|u\|_{H^{\frac{3}{2}+\nu}(\Omega)^d} + \|p\|_{H^{\frac{3}{2}+\nu}(\Omega)}).$$

We denote by $\Upsilon = \bigcup_{i=1}^N (\omega_i \cap \Omega)$. Let $r > 0$. Let us consider a finite cover of $\Omega\backslash\overline{\Upsilon}$: there exist $\tilde{N} \in \mathbb{N}$ and $y_i \in \Omega$, $i = 1, \ldots, \tilde{N}$, such that $\Omega\backslash\overline{\Upsilon} \subset \bigcup_{i=1}^{\tilde{N}} B(y_i, r)$. For all $i = 1, \ldots, \tilde{N}$, up to a decreasing r, $B(y_i, r)$ is a relatively compact open set in Ω where we can apply inequality (2.5) of proposition 2.4: there exist $c, s > 0$, such that for all $i \in \{1, \ldots, \tilde{N}\}$, for all $\epsilon > 0$, for all $(u, p) \in H^1(\Omega)^d \times H^1(\Omega)$ solutions of (1.1),

$$\|u\|_{H^1(\mathcal{B}(y_i,r))^d} + \|p\|_{H^1(\mathcal{B}(y_i,r))} \leqslant \frac{c}{\epsilon}(\|u\|_{H^1(\hat{\omega})^d} + \|p\|_{H^1(\hat{\omega})}) + \epsilon^s(\|u\|_{H^1(\Omega)^d} + \|p\|_{H^1(\Omega)}).$$

Thus, by summing up the two previous inequalities, taking into account remark 2.13, we obtain that for all $\beta \in \left(0, \frac{1}{2} + \nu\right)$, there exists $c > 0$, such that for all $\epsilon > 0$, for all $(u, p) \in H^{\frac{3}{2}+\nu}(\Omega)^d \times H^{\frac{3}{2}+\nu}(\Omega)$ solutions of (1.1),

$$\|u\|_{H^1(\Omega)^d} + \|p\|_{H^1(\Omega)} \leqslant \mathrm{e}^{\frac{c}{\epsilon}}(\|u\|_{H^1(\hat{\omega})^d} + \|p\|_{H^1(\hat{\omega})}) + \epsilon^\beta(\|u\|_{H^{\frac{3}{2}+\nu}(\Omega)^d} + \|p\|_{H^{\frac{3}{2}+\nu}(\Omega)}). \quad (2.26)$$

It remains to pass from a relatively compact open set $\hat{\omega}$ to an open set ω (not necessarily relatively compact): we use inequality (2.5) of proposition 2.4 in order to estimate $\|u\|_{H^1(\hat{\omega})^d} + \|p\|_{H^1(\hat{\omega})}$ in inequality (2.26) by $\|u\|_{H^1(\omega)^d} + \|p\|_{H^1(\omega)}$. It directly gives us inequality (2.4) of theorem 2.3.

Now, if we apply proposition 2.6, we obtain, ϵ being suitably chosen,

$$\|u\|_{H^1(\Omega)^d} + \|p\|_{H^1(\Omega)} \leqslant \mathrm{e}^{\frac{c}{\epsilon}} \left(\|u\|_{H^1(\Gamma)^d} + \|p\|_{H^1(\Gamma)} + \left\| \frac{\partial u}{\partial n} \right\|_{L^2(\Gamma)^d} + \left\| \frac{\partial p}{\partial n} \right\|_{L^2(\Gamma)} \right)$$

$$+ \epsilon^{\beta} (\|u\|_{H^{\frac{3}{2}+\nu}(\Omega)^d} + \|p\|_{H^{\frac{3}{2}+\nu}(\Omega)}). \qquad (2.27)$$

Let $\theta = \frac{1}{1+\nu} \in (0, 1)$. Using an interpolation inequality, we obtain that there exists $c > 0$ such that

$$\|u\|_{H^1(\Gamma)^d} + \|p\|_{H^1(\Gamma)} \leqslant c\big(\|u\|_{L^2(\Gamma)^d}^{1-\theta} \|u\|_{H^{1+\nu}(\Gamma)^d}^{\theta} + \|p\|_{L^2(\Gamma)}^{1-\theta} \|p\|_{H^{1+\nu}(\Gamma)}^{\theta}\big).$$

If we write that

$$\|u\|_{L^2(\Gamma)^d}^{1-\theta} \|u\|_{H^{1+\nu}(\Gamma)^d}^{\theta} = \mathrm{e}^{\frac{2c\theta}{\epsilon}} \|u\|_{L^2(\Gamma)^d}^{1-\theta} \, \mathrm{e}^{-\frac{2c\theta}{\epsilon}} \|u\|_{H^{1+\nu}(\Gamma)^d}^{\theta}$$

and

$$\|p\|_{L^2(\Gamma)}^{1-\theta} \|p\|_{H^{1+\nu}(\Gamma)}^{\theta} = \mathrm{e}^{\frac{2c\theta}{\epsilon}} \|p\|_{L^2(\Gamma)}^{1-\theta} \, \mathrm{e}^{-\frac{2c\theta}{\epsilon}} \|p\|_{H^{1+\nu}(\Gamma)}^{\theta},$$

according to the Young inequality and to the continuity of the trace operator from $H^{\frac{3}{2}+\nu}(\Omega)$ onto $H^{1+\nu}(\Gamma)$, we obtain

$$\|u\|_{H^1(\Gamma)^d} + \|p\|_{H^1(\Gamma)} \leqslant c(\mathrm{e}^{\frac{-2c}{\epsilon}} (\|u\|_{H^{\frac{3}{2}+\nu}(\Omega)^d} + \|p\|_{H^{\frac{3}{2}+\nu}(\Omega)}) + \mathrm{e}^{\frac{2c}{\epsilon\nu}} (\|u\|_{L^2(\Gamma)^d} + \|p\|_{L^2(\Gamma)})).$$

Using the fact that $\mathrm{e}^{\frac{-c}{\epsilon}} \leqslant C\epsilon^{\beta}$ for all $\epsilon > 0$ allows us to replace $\|u\|_{H^1(\Gamma)^d} + \|p\|_{H^1(\Gamma)}$ on the right-hand side of inequality (2.27) by $\|u\|_{L^2(\Gamma)^d} + \|p\|_{L^2(\Gamma)}$. This proves inequality (2.3) of theorem 2.3. □

Proof of theorem 2.1. We introduce two open sets $\tilde{\Omega}$ and $\tilde{\omega} \subset \tilde{\Omega}$ such that $\tilde{\omega} \subset\subset \omega$ and $K \subset\subset \tilde{\Omega} \subset\subset \Omega$, Let us apply proposition 2.4 to estimate u and p on K :

$$\begin{cases} \exists\, c, s > 0, \forall\, \epsilon > 0, \forall\, (u, p) \in H^1(\Omega)^d \times H^1(\Omega) \text{ solutions of } (1.1), \\ \|u\|_{H^1(K)^d} + \|p\|_{L^2(K)} \leqslant \dfrac{c}{\epsilon}(\|u\|_{H^1(\tilde{\omega})^d} + \|p\|_{H^1(\tilde{\omega})}) + \epsilon^s(\|u\|_{H^1(\tilde{\Omega})^d} + \|p\|_{H^1(\tilde{\Omega})}). \end{cases} \qquad (2.28)$$

Then, theorem 2.1 directly follows thanks to the Caccioppoli inequality that we recall in lemma 2.17 below (see [20]). □

Lemma 2.17 (Caccioppoli inequality)**.** *Let v be a weak solution of $\Delta v = 0$ in $\Omega \subset \mathbb{R}^d$. Then, there exists $C > 0$ such that for all $x_0 \in \Omega$ and $0 < \rho < R < d(x_0, \partial\Omega)$, we have*

$$\int_{B(x_0,\rho)} |\nabla v|^2 \leqslant \frac{C}{(R-\rho)^2} \int_{B(x_0,R)} |v|^2.$$

Remark 2.18. The proof of proposition 2.4 together with the Caccioppoli inequality contains all the tools needed to prove an interesting result, which is, in the case of the Stokes system, a three-balls inequality involving the velocity in the H^1-norm and the pressure in the L^2-norm. We refer to [16] for a statement and a proof of this result.

2.4. Comments

Let us now conclude this section by some comments. By borrowing the approach developed by Phung in [25], we have thus proved the stability estimates stated in theorems 2.1 and 2.3 that quantify the unique continuation result of Fabre and Lebeau in [18]. The Carleman estimate that we use near the boundary is a consequence of pseudo-differential calculus. To apply this technique, the domain has to be very regular. In [8], Bourgeois proved that the stability estimates proved by Phung in [25] for $\mathcal{C}^\infty$ domains still hold for domains of class $\mathcal{C}^{1,1}$. To derive estimates near the boundary, he used the global Carleman estimate near the boundary on the initial geometry, by following the method of [19]. Moreover, in [9], Bourgeois and Dardé completed the results obtained in [8]: they proved a conditional stability estimate related to the ill-posed Cauchy problem for the Laplace equation in domain with the Lipschitz boundary. For such non-smooth domains, difficulties occur when one wants to estimate the function in a neighborhood of $\partial\Omega$: the authors use an interior Carleman estimate and a technique based on a sequence of balls which approach the boundary, which is inspired by [1]. Let us emphasize the fact that the inequality obtained in this way is valid for a regular solution u (u belongs to $\mathcal{C}^{1,\alpha}(\Omega)$ and is such that $\Delta u \in L^2(\Omega)$), and that boundary conditions are known on a part of the boundary. These two results suggest that it could be possible to extend estimates (2.3) and (2.4) to less regular open sets. Another improvement could be to study whether the stability estimate of proposition 2.6 still holds if we have less measurements on the boundary. Let us note that in a recent work, one of the authors obtains a Lipschitz stability estimate under the *a priori* assumption that the Robin coefficient lives in some compact and convex subset of a finite-dimensional vectorial subspace of the set of continuous functions involving only the velocity in the L^2-norm (see [14]).

These kinds of stability estimates can be used for different purposes. For example, Phung uses the stability estimate stated in [25] for the Laplace equation to establish an estimate of the cost of an approximate control function for an elliptic model equation. In [9], Bourgeois and Dardé use stability estimates to study the convergence rate for the method of quasi-reversibility introduced in [22] to solve Cauchy problems. In [5], Bellassoued *et al* use a stability estimate proved by Phung in [25] to solve an inverse problem similar to the one we are interested in, but for the Laplace equation. As far as we are concerned, we are going to use our stability estimates to study the inverse problem of identifying a Robin coefficient from measurements available on a part of the boundary in the Stokes system: this is the subject of the next section.

3. Application to an inverse problem

Throughout this section, we assume that the boundary of Ω is composed of two sets Γ_0 and Γ_{out} such that $\Gamma_{\text{out}} \cup \Gamma_0 = \partial\Omega$ and $\overline{\Gamma}_{\text{out}} \cap \overline{\Gamma}_0 = \emptyset$. An example of such geometry in dimension 2 is given in figure 1.

Let us recall the inverse problem we are interested in: we want to obtain a stability result for the Robin coefficient q defined on Γ_{out} with respect to the values of u and p on $\Gamma \subseteq \Gamma_0$ for the (u, p) solution of system (1.5). Let us point out that the uniqueness issue related to our inverse problem has already been studied in [7] and is a consequence of corollary 1.2. More precisely, in [7], we have stated that, under some assumptions on the flux g and on the Robin coefficient q, if the velocities are equal on some nonempty open set $\Gamma \subseteq \Gamma_0$, then the Robin coefficients are equal on Γ_{out}.

Notation 3.1. *We introduce the following functional spaces:*

$$V = \{v \in H^1(\Omega)^d \mid \operatorname{div} v = 0 \text{ } on \text{ } \Omega\}$$

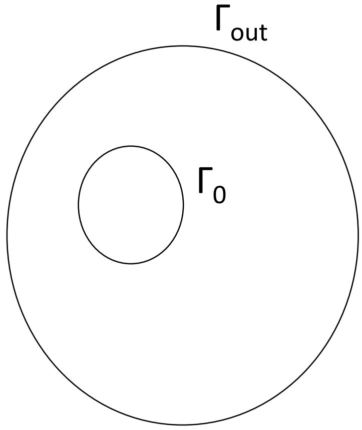

Figure 1. Example of an open set $\Omega \subset \mathbb{R}^2$, such that $\partial\Omega = \Gamma_0 \cup \Gamma_{out}$ and $\overline{\Gamma}_0 \cap \overline{\Gamma}_{out} = \emptyset$.

and

$$H = \overline{V}^{L^2(\Omega)^d}.$$

Before proving theorem 1.5, let us recall the following regularity result (which is proved in [7]).

Proposition 3.2. *Let $k \in \mathbb{N}$ and $s \in \mathbb{R}$ be such that $s > \frac{d-1}{2}$ and $s \geqslant \frac{1}{2} + k$. Assume that Ω is of class $C^{k+1,1}$. Let $\alpha > 0$, $M > 0$, $g \in H^{\frac{1}{2}+k}(\Gamma_0)^d$ and $q \in H^s(\Gamma_{out})$ be such that $\alpha \leqslant q$ on Γ_{out}. Then, the solution (u, p) of system (1.5) belongs to $H^{k+2}(\Omega)^d \times H^{k+1}(\Omega)$. Moreover, there exists a constant $C(\alpha, M) > 0$ such that for every $q \in H^s(\Gamma_{out})$ satisfying $\|q\|_{H^s(\Gamma_{out})} \leqslant M$,*

$$\|u\|_{H^{k+2}(\Omega)^d} + \|p\|_{H^{k+1}(\Omega)} \leqslant C(\alpha, M)\|g\|_{H^{k+\frac{1}{2}}(\Gamma_0)^d}.$$

Proof of theorem 1.5. Let us emphasize the fact that, thanks to proposition 3.2, there exists $C(\alpha, M_1, M_2) > 0$ such that

$$\|u_j\|_{H^{k+2}(\Omega)^d} + \|p_j\|_{H^{k+1}(\Omega)} \leqslant C(\alpha, M_1, M_2) \quad \text{for } j = 1,\ 2. \tag{3.1}$$

In the following, we denote by $u = u_1 - u_2$ and $p = p_1 - p_2$. We have

$$(q_2 - q_1)u_1 = q_2 u + \frac{\partial u}{\partial n} - pn \quad \text{on } \Gamma_{out}. \tag{3.2}$$

Consequently, since $|u_1| \geqslant m > 0$ on K,

$$\|q_1 - q_2\|_{L^2(K)} \leqslant \frac{1}{m} C(M_2) \left(\|u\|_{L^2(K)^d} + \left\| \frac{\partial u}{\partial n} \right\|_{L^2(K)^d} + \|p\|_{L^2(K)} \right). \tag{3.3}$$

Since K and Γ are in $\mathcal{C}^\infty$, we can construct an open set $\omega \subset \Omega$ of class $\mathcal{C}^\infty$ such that $K \subset \partial\omega$ and $\Gamma \subset \partial\omega$. Then, for all $0 < \epsilon < \frac{3}{2}$, using the trace continuity and an interpolation inequality, we have

$$\begin{aligned} \|q_1 - q_2\|_{L^2(K)} &\leqslant \frac{1}{m} C(M_2)(\|u\|_{H^{3/2+\epsilon}(\omega)^d} + \|p\|_{L^2(\omega)}) \\ &\leqslant \frac{1}{m} C(M_2)\big(\|u\|^{\theta}_{H^1(\omega)^d} \|u\|^{1-\theta}_{H^3(\omega)^d} + \|p\|_{L^2(\omega)}\big), \end{aligned} \tag{3.4}$$

where $\theta = \frac{3}{4}\left(1 - \frac{2\epsilon}{3}\right)$. According to inequality (3.1), we then deduce

$$\|q_1 - q_2\|_{L^2(K)} \leqslant \frac{1}{m} C(\alpha, M_1, M_2)\left(\|u\|^{\theta}_{H^1(\omega)^d} + \|p\|^{\theta}_{L^2(\omega)}\right).$$

Let $\beta \in (0, 1)$ be fixed. We choose $0 < \epsilon < \frac{3}{2}$ to be small enough such that $\beta' = \frac{\beta}{1-\frac{2\epsilon}{3}}$ belongs to $(0, 1)$. We denote by $A = \|u\|_{H^2(\omega)^d} + \|p\|_{H^2(\omega)}$ and $B = \|u\|_{L^2(\Gamma)^d} + \|p\|_{L^2(\Gamma)} + \left\|\frac{\partial u}{\partial n}\right\|_{L^2(\Gamma)^d} + \left\|\frac{\partial p}{\partial n}\right\|_{L^2(\Gamma)}$. Applying inequality (1.3) of theorem 1.4 with $\nu = \frac{1}{2}$ and with β', we find that there exists $d_0 > 0$ such that for all $\tilde{d} > d_0$, there exists $C(\alpha, M_1, M_2) > 0$:

$$\|q_1 - q_2\|_{L^2(K)} \leqslant \frac{1}{m} C(\alpha, M_1, M_2) \frac{A^{\theta}}{\left(\ln\left(\tilde{d}\frac{A}{B}\right)\right)^{\beta'\theta}}. \tag{3.5}$$

We conclude by studying the variation of the function defined by $f_y(x) = \frac{x}{(\ln(\frac{x}{y}))^{\beta'}}$ on $(y, +\infty)$ for $y = \frac{B}{\tilde{d}}$. We have $f'_y(x) = \frac{\ln(\frac{x}{y})-\beta'}{(\ln(\frac{x}{y}))^{\beta'+1}}$. Let us denote by $x_0 = y\,\mathrm{e}^{\beta'}$. The function f_y is decreasing on $(y, x_0]$ and is increasing on $[x_0, +\infty)$. For $\tilde{d}$ being large enough, $A \geqslant x_0$. Thanks to (3.1) and since f is increasing on $[x_0, +\infty)$, we directly deduce that $f_{\frac{B}{\tilde{d}}}(A) \leqslant f_{\frac{B}{\tilde{d}}}(C(\alpha, M_1, M_2))$. Using this result in (3.5), we find that there exist $C(\alpha, M_1, M_2) > 0$ and $C_1(\alpha, M_1, M_2) > 0$, such that

$$\|q_1 - q_2\|_{L^2(K)} \leqslant \frac{1}{m} \frac{C(\alpha, M_1, M_2)}{\left(\ln\left(\frac{C_1(\alpha, M_1, M_2)}{\|u\|_{L^2(\Gamma)^d} + \|p\|_{L^2(\Gamma)} + \|\frac{\partial u}{\partial n}\|_{L^2(\Gamma)^d} + \|\frac{\partial p}{\partial n}\|_{L^2(\Gamma)}}\right)\right)^{\beta'\theta}},$$

and since $\beta'\theta = 3\beta/4$ and $\frac{\partial u}{\partial n} = pn$ on Γ, this concludes the proof of the theorem. □

Remark 3.3. Since g is not identically zero, corollary 1.2 ensures that $\{x \in \Gamma_0 \mid u_1(x) \neq 0\}$ is not empty. Moreover, according to proposition 3.2, u_1 is continuous; thus, we obtain the existence of a compact K and a constant m as in theorem 1.5. We note however that the constants involved in estimate (1.6) and the set K depend on u_1. Finding a uniform lower bound for any solution u of system (1.5) remains an open question. We refer to [2], [3] and [11] for the case of the scalar Laplace equation.

Remark 3.4. Outside the set K, an estimate of $q_1 - q_2$ may be undetermined or highly unstable. In particular, an estimate of the Robin coefficients on the whole set Γ_{out} might be worse than that of the logarithmic type (see [6]). Note however that for a simplified problem, it is in fact possible to obtain a logarithmic stability estimate on the whole set Γ_{out} which does not depend on a given reference solution (see [7]).

Remark 3.5. In inequality (1.6), the power $3\beta/4$ is directly linked to the regularity of the solution (u, p). If we are more precise in our estimates, we can note that this power may be improved by a power which depends on k. Indeed, coming back to inequalities (3.4) and using that $(u, p) \in H^{k+2}(\Omega)^d \times H^{k+1}(\Omega)$, we obtain that

$$\|q_1 - q_2\|_{L^2(K)} \leqslant \frac{1}{m} C(M_2)\left(\|u\|^{\tilde{\theta}}_{H^1(\omega)^d}\|u\|^{1-\tilde{\theta}}_{H^{k+2}(\omega)^d} + \|p\|_{L^2(\omega)}\right),$$

where $\tilde{\theta} = \frac{1/2+k}{1+k} - \frac{\epsilon}{1+k}$. This estimate allows us to obtain the power $\frac{1/2+k}{1+k}\beta$ instead of $3\beta/4$ in inequality (1.6) (when $k = 1$, these powers are equal).

Remark 3.6. Note that we can still obtain inequality (1.6) by enforcing less regularity on the solution (u, p). In particular, if we consider the case when $d \leqslant 5$, it is sufficient to assume that (u_j, p_j) belongs to $H^{\frac{5}{2}+\nu}(\Omega) \times H^{\frac{3}{2}+\nu}(\Omega)$ and q_j belongs to $L^\infty(\Gamma_{\text{out}})$, and that

$$\|u_j\|_{H^{\frac{5}{2}+\nu}(\Omega)^d} + \|p_j\|_{H^{\frac{3}{2}+\nu}(\Omega)} \leqslant M_1 \text{ and } \|q_j\|_{L^\infty(\Gamma_{\text{out}})} \leqslant M_2$$

for $j = 1, 2$. In this case, the velocity u_1 is still continuous, and with the same reasons as in remark 3.3, there exist a compact K and a constant $m > 0$ as in theorem 1.5. Next, instead of inequalities (3.4), we use

$$\|q_1 - q_2\|_{L^2(K)} \leqslant \frac{1}{m} C(M_2)(\|u\|_{H^{3/2+\nu/3}(\omega)^d} + \|p\|_{L^2(\omega)})$$
$$\leqslant \frac{1}{m} C(M_2)\big(\|u\|^{2/3}_{H^1(\omega)^d}\|u\|^{1/3}_{H^{5/2+\nu}(\omega)^d} + \|p\|_{L^2(\omega)}\big).$$

Then, by applying the same reasoning as above, we find that for all $\beta \in (0, 1)$, there exist $C(\alpha, M_1, M_2) > 0$ and $C_1(\alpha, M_1, M_2) > 0$ such that

$$\|q_1 - q_2\|_{L^2(K)} \leqslant \frac{1}{m} \frac{C(\alpha, M_1, M_2)}{\left(\ln\left(\frac{C_1(\alpha, M_1, M_2)}{\|u_1 - u_2\|_{L^2(\Gamma)^d} + \|p_1 - p_2\|_{L^2(\Gamma)} + \|\frac{\partial p_1}{\partial n} - \frac{\partial p_2}{\partial n}\|_{L^2(\Gamma)}}\right)\right)^{\frac{2}{3}\beta}}.$$

Let us note that, due to the fact that the solution is less regular, the power in this inequality is weaker than that in inequality (1.6) ($2\beta/3$ instead of $3\beta/4$ for $\beta \in (0, 1)$).

Remark 3.7. Assume that $\partial\Omega = \Gamma_0 \cup \Gamma_{\text{out}} \cup \Gamma_l$, $\overline{\Gamma}_0 \cap \overline{\Gamma}_{\text{out}} = \emptyset$, $\overline{\Gamma}_l \cap \overline{\Gamma}_{\text{out}} = \emptyset$ and $\overline{\Gamma}_0 \cap \overline{\Gamma}_l = \emptyset$. Then, theorem 1.5 remains true for the (u, p) solution of the system

$$\begin{cases} -\Delta u + \nabla p = 0, & \text{in } \Omega, \\ \operatorname{div} u = 0, & \text{in } \Omega, \\ u = 0, & \text{on } \Gamma_l, \\ \dfrac{\partial u}{\partial n} - pn = g, & \text{on } \Gamma_0, \\ \dfrac{\partial u}{\partial n} - pn + qu = 0, & \text{on } \Gamma_{\text{out}}, \end{cases}$$

where we have added a homogeneous Dirichlet boundary condition on the part of the boundary Γ_l. Indeed, for this problem, we still have enough regularity on the solution to apply the same reasoning as in the proof of theorem 1.5. Moreover, we can discard the assumptions $\overline{\Gamma}_l \cap \overline{\Gamma}_{\text{out}} = \emptyset$ and $\overline{\Gamma}_0 \cap \overline{\Gamma}_l = \emptyset$ if we assume that $K \subset\subset \Gamma_0$ and $\Gamma \subset\subset \Gamma_{\text{out}}$. This allows us to consider domains which are closer to the ones that we encounter in applications such as, for instance, the first generations of the bronchial tree (see [4]).

Remark 3.8. As in [7], we can obtain from theorem 1.5 a stability estimate for the unsteady problem when the Robin coefficient does not depend on time and under assumptions on the asymptotic behavior of the flux g when it depends on time. The key idea is to estimate the difference between the solution of the stationary problem and the solution of the non-stationary problem by a function which tends to zero as t tends to zero, using an inequality coming from semigroup theory. Doing so, the measurements have to be made in infinite time. Let us recall that Bellassoued *et al* have already used this idea in [5] in the case of the Laplace equation with mixed Neumann and Robin boundary conditions.

The result stated in theorem 1.5 could be improved in different ways. In the stability estimate (1.6), the Robin coefficients are estimated on a compact subset $K \subset \Gamma_{\text{out}}$ which is not a fixed inner portion of Γ_{out}, but is unknown and depends on a given reference solution. To the best of our knowledge, obtaining an estimate of the Robin coefficients on the whole set Γ_{out} or on any compact subset $K \subset \Gamma_{\text{out}}$ is still an open question. At last, another natural issue concerns the optimality of the logarithmic stability estimates. In [29], the Hölder stability estimates are obtained for the scalar Laplace equation when the Robin coefficient is piecewise constant. For the Stokes equation, this question has been addressed in [16] where a similar result has been proved.

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References

[1] Alessandrini G, Beretta E, Rosset E and Vessella S 2000 Optimal stability for inverse elliptic boundary value problems with unknown boundaries *Ann. Scuola Norm. Super. Pisa Cl. Sci.* **29** 755–806
[2] Alessandrini G, Del Piero L and Rondi L 2003 Stable determination of corrosion by a single electrostatic boundary measurement *Inverse Problems* **19** 973–84
[3] Alessandrini G and Sincich E 2006 Detecting nonlinear corrosion by electrostatic measurements *Appl. Anal.* **85** 107–28
[4] Baffico L, Grandmont C and Maury B 2010 Multiscale modeling of the respiratory tract *Math. Models Methods Appl. Sci.* **20** 59–93
[5] Bellassoued M, Cheng J and Choulli M 2008 Stability estimate for an inverse boundary coefficient problem in thermal imaging *J. Math. Anal. Appl.* **343** 328–36
[6] Bellassoued M, Choulli M and Jbalia A 2012 Stability of the determination of the surface impedance of an obstacle from the scattering amplitude doi:10.1002/mma.2762
[7] Boulakia M, Egloffe A-C and Grandmont C 2013 Stability estimates for a Robin coefficient in the two-dimensional Stokes system *Math. Control Relat. Field* **3** 21–49
[8] Bourgeois L 2010 About stability and regularization of ill-posed elliptic Cauchy problems: the case of $C^{1,1}$ domains *Math. Modelling Numer. Anal.* **44** 715–35
[9] Bourgeois L and Dardé J 2010 About stability and regularization of ill-posed elliptic Cauchy problems: the case of Lipschitz domains *Appl. Anal.* **89** 1745–68
[10] Chaabane S, Fellah I, Jaoua M and Leblond J 2004 Logarithmic stability estimates for a Robin coefficient in two-dimensional Laplace inverse problems *Inverse Problems* **20** 47–59
[11] Chaabane S and Jaoua M 1999 Identification of Robin coefficients by the means of boundary measurements *Inverse Problems* **15** 1425–38
[12] Cheng J, Choulli M and Lin J 2008 Stable determination of a boundary coefficient in an elliptic equation *Math. Models Methods Appl. Sci.* **18** 107–23
[13] Dautray R and Lions J-L (With the Collaboration of Bénilan P, Cessenat M, Gervat A, Kavenoky A and Lanchon H) 1990 *Mathematical Analysis and Numerical Methods for Science and Technology: Physical Origins and Classical Methods* vol 1 (Berlin: Springer) (Translated from French by Ian N Sneddon, with a preface by Jean Teillac)
[14] Egloffe A-C 2013 Lipschitz stability estimate in the inverse Robin problem for the Stokes system *C. R. Acad. Sci., Paris* at press
[15] Egloffe A-C 2012 Étude de quelques problèmes inverses pour le système de Stokes. Application aux poumons *PhD Thesis* Université Paris VI
[16] Egloffe A-C 2013 Lipschitz stability estimate in the inverse Robin problem for the Stokes system *INRIA Research Report* RR-8222 (INRIA)
[17] Lions J L and Magenes E 1968 *Problèmes aux Limites Non Homogènes et Applications* vol 1 (Paris: Dunod)
[18] Fabre C and Lebeau G 1996 Prolongement unique des solutions de l'equation de Stokes *Commun. Partial Differ. Eqns* **21** 573–96
[19] Fursikov A V and Imanuvilov O Y 1996 *Controllability of Evolution Equations* (*Lecture Notes Series* vol 34) (Seoul: Global Analysis Research Center, Research Institute of Mathematics, Seoul National University)
[20] Giaquinta M and Souček J 1985 Caccioppoli's inequality and Legendre–Hadamard condition *Math. Ann.* **270** 105–7
[21] Hörmander L 1985 *The Analysis of Linear Partial Differential Operators: III. Pseudo-Differential Operators* (*Fundamental Principles of Mathematical Sciences* vol 274) (Berlin: Springer)
[22] Lattès R and Lions J-L 1967 *Méthode de Quasi-réversibilité et Applications* (*Travaux et Recherches Mathématiques* vol 15) (Paris: Dunod)
[23] Lebeau G and Robbiano L 1995 Contrôle exact de l'équation de la chaleur *Commun. Partial Differ. Eqns* **20** 335–56
[24] Lin C-L, Uhlmann G and Wang J-N 2010 Optimal three-ball inequalities and quantitative uniqueness for the Stokes system *Discrete Contin. Dyn. Syst.* **28** 1273–90

[25] Phung K-D 2003 Remarques sur l'observabilité pour l'équation de Laplace *ESAIM: Control, Optim. Calculus Variations* **9** 621–35

[26] Quarteroni A and Veneziani A 2003 Analysis of a geometrical multiscale model based on the coupling of ODEs and PDEs for blood flow simulations *Multiscale Modeling Simul.* **1** 173–95

[27] Robbiano L 1991 Théorème d'unicité adapté au contrôle des solutions des problèmes hyperboliques *Commun. Partial Differ. Eqns* **16** 789–800

[28] Robbiano L 1995 Fonction de coût et contrôle des solutions des équations hyperboliques *Asymptotic Anal.* **10** 95–115

[29] Sincich E 2007 Lipschitz stability for the inverse Robin problem *Inverse Problems* **23** 1311–26

[30] Vignon-Clementel I E, Figueroa C A, Jansen K E and Taylor C A 2006 Outflow boundary conditions for three-dimensional finite element modeling of blood flow and pressure in arteries *Comput. Methods Appl. Mech. Eng.* **195** 3776–96