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# Existence of weak solutions for an interaction problem between an elastic structure and a compressible viscous fluid

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## Abstract

In this article, we are interested by the three-dimensional motion of an elastic structure immersed in a viscous compressible fluid. The fluid and the structure are contained in a fixed bounded set. To describe the structure motion, we choose an Eulerian point of view and we strongly regularize the equation of the solid motion in order to get additional estimates on the elastic deformations. Our main result is an existence result of weak solutions defined as long as no collisions occur and as long as conditions of non-interpenetration and of preservation of orientation are satisfied.

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## Résumé

Dans cet article, nous étudions le mouvement d'une structure élastique immergée dans un fluide compressible en dimension trois. Le fluide et la structure sont contenus dans une cavité fixe bornée. On prend un point de vue eulérien pour décrire le mouvement de la structure et les équations du mouvement solide sont fortement régularisées afin d'obtenir des estimations supplémentaires sur les déformations élastiques. Notre principal résultat est un résultat d'existence de solutions faibles définies tant qu'il n'y a pas de chocs entre la structure et la paroi de la cavité et tant que des conditions de non-interpénétration et de préservation de l'orientation du solide sont satisfaites.

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*Keywords:* Fluid–structure interaction; Navier–Stokes equations; Compressible fluid; Existence of weak solutions

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## 1. Introduction and equations of motion

In this paper, we consider the motion of an elastic structure immersed in a viscous compressible fluid described by the compressible Navier–Stokes equations. The fluid and the structure are contained in a fixed bounded set  $\Omega \subset \mathbb{R}^3$  which is supposed to be regular enough. We consider regularized elastic deformations for the structure and we prove an existence result of weak solutions for this problem. Solutions are defined as long as there is no collision and as long as conditions of non-interpenetration and of preservation of orientation are satisfied by the displacement field of the structure.

For related works on models dealing with an elastic structure and an incompressible fluid, we refer to [2,5,8,10] (see also references therein). The case of rigid structures immersed in a compressible fluid is treated in [9]. The problem of interaction between a compressible fluid and an elastic plate occupying a part of the fluid domain boundary is considered in [14] and [15]. In these works, the fluid motion is modelled by an equation which is linear in the velocity (the convective term is not considered). To the best of our knowledge, we present in this paper the first existence result dealing with the interaction between a compressible fluid modelled by the Navier–Stokes equation and an elastic structure.

To show our existence result, we follow the method introduced in the article [13] which proves the global existence of weak solutions to the compressible Navier–Stokes equations. This paper improves the existence result obtained in [19] which gives the first existence result for compressible fluids without restrictions on the initial conditions or geometry of the domain. The method presented in [13] has already been adapted to the case of a rigid structure immersed in a compressible fluid in [12].

We denote by  $\Omega_S(t)$  the domain occupied by the structure and  $\Omega_F(t) = \Omega \setminus \overline{\Omega_S(t)}$  the domain occupied by the fluid at time  $t$ . The fluid motion is governed by the compressible Navier–Stokes equations:

$$\partial_t(\varrho_F u_F) + \operatorname{div}(\varrho_F u_F \otimes u_F) + \nabla p - \operatorname{div} \mathbb{T} = 0 \quad \text{in } \Omega_F(t), \quad (1.1)$$

where  $u_F$  denotes the Eulerian velocity,  $p$  the pressure and  $\varrho_F$  the density. The stress tensor  $\mathbb{T}$  is defined by:

$$\mathbb{T} = \mu_F \nabla u_F + (\lambda_F + \mu_F) \operatorname{div} u_F \operatorname{Id},$$

where the viscosity coefficients  $\lambda_F$  and  $\mu_F$  are such that

$$\mu_F > 0, \quad 3\lambda_F + 2\mu_F \geq 0.$$

The pressure and the density are functionally dependent and the relation between them is given by the constitutive law:

$$p = a\varrho_F^\gamma,$$

where  $a$  is a strictly positive constant and  $\gamma > 3/2$  is the adiabatic constant. Moreover, the density  $\varrho_F$  satisfies the continuity equation:

$$\partial_t \varrho_F + \operatorname{div}(\varrho_F u_F) = 0 \quad \text{on } \Omega_F(t). \tag{1.2}$$

On the structure, we choose to keep this Eulerian point of view. We will see that this choice consequently simplifies the writing of the global problem. For instance, this allows to deal with test functions independent of the solution. Furthermore, as the Lagrangian flow solution of the problem will be invertible, this Eulerian formulation will be equivalent to a more usual Lagrangian formulation.

Let  $u_S$  be the Eulerian velocity of the structure,  $\varrho_S$  the density of the structure and  $X_S$  the Lagrangian flow. For all  $t$  in  $[0, T]$ , for all  $y$  in  $\Omega_S(0)$ ,  $X_S(t, 0, y)$  is the position at time  $t$  of the particle located in  $y$  at initial time. The relation between  $u_S$  and  $X_S$  is: for all  $y \in \Omega_S(0)$ ,

$$\begin{cases} \partial_t X_S(t, 0, y) = u_S(t, X_S(t, 0, y)), \\ X_S(0, 0, y) = y. \end{cases} \tag{1.3}$$

If  $u_S$  is enough regular (this will be satisfied by our solution),  $X_S$  is well defined and for each  $t \in (0, T)$ ,  $X_S(t, 0, \cdot)$  is invertible from  $\Omega_S(0)$  on  $\Omega_S(t)$ , we denote  $X_S(0, t, \cdot)$  the inverse. Next, we consider the following momentum equation:

$$\partial_t(\varrho_S u_S) + \operatorname{div}(\varrho_S u_S \otimes u_S) + \theta A_3 u_S - \operatorname{div} \sigma_S = 0 \quad \text{in } \Omega_S(t). \tag{1.4}$$

The term  $\theta A_3 u_S$  is a regularizing term; the regularizing parameter  $\theta$  is a fixed strictly positive real number and  $A_3$  is the differential operator defined by: for all  $l = 1, 2, 3$ , for all  $u$  regular enough,

$$(A_3 u)_l = -\frac{1}{2} \sum_{i=1}^3 \frac{\partial \varepsilon_{i,l}(u)}{\partial x_i} + \sum_{i,j=1}^3 \frac{\partial^4 u_l}{\partial x_i^2 \partial x_j^2} - \sum_{i,j,k=1}^3 \frac{\partial^6 u_l}{\partial x_i^2 \partial x_j^2 \partial x_k^2},$$

where  $\varepsilon(u)$  denotes the symmetric part of the gradient of  $u$ .

Thus, we have:  $\forall u, v \in \mathcal{D}(\Omega_S(t))^3$ ,

$$\int_{\Omega_S(t)} A_3 u v = ((u, v))_{H^3(\Omega_S(t))},$$

where we have defined:  $\forall u, v \in H^3(\Omega_S(t))^3$ ,

$$\begin{aligned} ((u, v))_{H^3(\Omega_S(t))} &= \int_{\Omega_S(t)} \varepsilon(u) : \varepsilon(v) + \sum_{i,j=1}^3 \int_{\Omega_S(t)} \frac{\partial^2 u_l}{\partial x_i \partial x_j} \frac{\partial^2 v_l}{\partial x_i \partial x_j} \\ &+ \sum_{i,j,k=1}^3 \int_{\Omega_S(t)} \frac{\partial^3 u_l}{\partial x_i \partial x_j \partial x_k} \frac{\partial^3 v_l}{\partial x_i \partial x_j \partial x_k}. \end{aligned}$$

Thanks to this regularization, the flow  $X_S$  will belong to  $H^1(0, T; H^3(\Omega_S(0)))$ . We can notice, that if we only consider a rigid velocity on the structure,  $A_3$  does not act on it.

**Remark 1.** Here, the abstract operator  $A_3$  has no physical meaning: this term is added because it is necessary to our study (we will explain later why we need this regularizing term). However, in the theory of multipolar materials (see [21]), stress tensors with spatial derivatives of high order are introduced with a physical interpretation: our regularizing term corresponds to a tripolar material.

The Cauchy stress tensor  $\sigma_S$  is expressed with respect to the second Piola–Kirchhoff tensor  $\hat{\sigma}_S$ :

$$\sigma_S(t, x) = \det \nabla X_S(0, t, x) \nabla X_S(0, t, x)^{-1} \hat{\sigma}_S(t, X_S(0, t, x)) \nabla X_S(0, t, x)^{-t},$$

$$\forall x \in \Omega_S(t),$$

and the constitutive law is the Saint-Venant–Kirchhoff law:

$$\hat{\sigma}_S[X_S] = 2\mu_S E(X_S) + \lambda_S \operatorname{tr}(E(X_S)) \operatorname{Id}, \quad (1.5)$$

where the Lamé constants of the elastic media  $\lambda_S$  and  $\mu_S$  satisfy:

$$\mu_S > 0, \quad \lambda_S + 2\mu_S > 0,$$

and  $E(X_S)$  is the Green–Saint-Venant tensor defined by:

$$E(X_S) = \frac{1}{2} ({}^t \nabla X_S \nabla X_S - \operatorname{Id}).$$

At last, the evolution of  $\varrho_S$  is given by the continuity equation:

$$\partial_t \varrho_S + \operatorname{div}(\varrho_S u_S) = 0 \quad \text{on } \Omega_S(t). \quad (1.6)$$

This system is completed by boundary conditions. As the fluid is viscous, the velocity is continuous at the interface:

$$\begin{cases} u_F = 0 & \text{on } \partial\Omega, \\ u_F = u_S & \text{on } \partial\Omega_S(t). \end{cases} \quad (1.7)$$

The second equation is a coupling equation between the fluid and the structure. The coupling is also expressed by the continuity of the stress on the interface: for all  $t \in [0, T]$  and for all  $v \in \mathcal{C}(\partial\Omega_S(t))$ ,

$$\int_{\partial\Omega_S(t)} (\mathbb{T} - p \text{Id})n_x \cdot v = \int_{\partial\Omega_S(t)} \sigma_S n_x \cdot v - \theta \langle u_S, v \rangle_{3, \partial\Omega_S(t)}, \tag{1.8}$$

where the operator  $\langle \cdot, \cdot \rangle_{3, \partial\Omega_S(t)}$  represents the contributing terms on the boundary of the regularizing operator  $A_3$ :  $\forall u, v \in \mathcal{D}(\overline{\Omega_S(t)})^3$ ,

$$\int_{\Omega_S(t)} A_3 uv = ((u, v))_{H^3(\Omega_S(t))} + \langle u, v \rangle_{3, \partial\Omega_S(t)}.$$

Moreover, the vector  $n_x$  is the outwards unit normal to  $\partial\Omega_S(t)$  at point  $x$ . We denote by  $u$  the global Eulerian velocity and by  $\varrho$  the global density defined on  $\Omega$ . Eqs. (1.2) and (1.6) are equivalent to,

$$\partial_t \varrho + \text{div}(\varrho u) = 0 \quad \text{in } \Omega. \tag{1.9}$$

At last, we prescribe initial data  $u^0$  in  $H_0^1(\Omega)$ ,  $\varrho_S^0$  in  $L^\infty(\Omega_S(0))$  and  $\varrho_F^0$  in  $L^\gamma(\Omega_F(0))$ :

$$u(t=0) = u^0 \quad \text{in } \Omega, \quad \varrho(t=0) = \varrho^0 = \begin{cases} \varrho_S^0 & \text{in } \Omega_S(0), \\ \varrho_F^0 & \text{in } \Omega_F(0). \end{cases} \tag{1.10}$$

Formally, the system given by equations (1.1)–(1.4) and (1.6) and boundary conditions (1.7) and (1.8) satisfies an a priori energy estimate:

$$\begin{aligned} & \frac{1}{2} \int_{\Omega} \varrho(t) |u(t)|^2 dx + \frac{a}{\gamma - 1} \int_{\Omega_F(t)} \varrho_F^\gamma(t) + \mu_F \int_0^t \int_{\Omega_F(s)} |\nabla u_F(s)|^2 \\ & + (\lambda_F + \mu_F) \int_0^t \int_{\Omega_F(s)} |\text{div} u_F(s)|^2 + \theta \int_0^t ((u_S(s), u_S(s)))_{H^3(\Omega_S(s))} \\ & + \mu_S \int_{\Omega_S(0)} |E(X_S(t, 0, y))|^2 dy + \frac{\lambda_S}{2} \int_{\Omega_S(0)} |\text{tr} E(X_S(t, 0, y))|^2 dy \leq E_0, \end{aligned} \tag{1.11}$$

where  $E_0$  is the initial energy,

$$E_0 = \frac{1}{2} \int_{\Omega} \varrho^0 |u^0|^2 dx + \frac{a}{\gamma - 1} \int_{\Omega_F(0)} (\varrho_F^0)^\gamma.$$

This comes in particular from the following calculation:

$$\begin{aligned} \int_{\Omega_S(t)} \sigma_S : \nabla u_S \, dx &= \int_{\Omega_S(0)} \hat{\sigma}_S(t, y) : \left[ {}^t \nabla X_S(t, 0, y) \nabla_y (u_S(t, X_S(t, 0, y))) \right] dy \\ &= \int_{\Omega_S(0)} \hat{\sigma}_S(t, y) : \partial_t E(X_S(t, 0, y)) \, dy. \end{aligned}$$

It is interesting to notice that if we choose the linearized elasticity law, the global system does not satisfy an energy estimate. Next, we define the concept of *renormalized solutions* introduced in [11] with slightly modified conditions on the admissible functions  $b$ :

**Definition 1.** The continuity equation (1.9) is satisfied in the sense of *renormalized solutions* if, for any  $b \in C^1(\mathbb{R})$  such that

$$b'(z) = 0 \quad \text{for } z \text{ large enough,} \quad (1.12)$$

we have:

$$\partial_t b(\varrho) + \operatorname{div}(b(\varrho)u) + (b'(\varrho)\varrho - b(\varrho)) \operatorname{div} u = 0 \quad \text{in } \mathcal{D}'((0, T) \times \Omega). \quad (1.13)$$

**Remark 2.** The condition (1.12) on the admissible functions can be weakened. Indeed, thanks to Lebesgue convergence theorem, we deduce that if (1.9) is satisfied in the sense of renormalized solutions for  $\varrho$  belonging to  $L^\infty(0, T; L^\alpha(\Omega))$  with  $\alpha > 3/2$ , then (1.13) holds for any  $b$  in  $C^1(\mathbb{R}_*^+) \cap C(\mathbb{R}^+)$  such that

$$|b'(z)z| \leq C(z^{\alpha/2} + z^\theta), \quad \forall z > 0 \text{ with } \theta < \frac{\alpha}{2}. \quad (1.14)$$

**Remark 3.** We assume that the adiabatic constant  $\gamma$  is greater than  $3/2$ . This condition is crucial in works dealing with compressible fluids. For instance, we can notice that  $\gamma = 3/2$  is the critical value for which the convective term is defined almost everywhere. Indeed, if  $\varrho$  belongs to  $L^\infty(0, T; L^\gamma(\Omega))$ , as  $u$  belongs to  $L^2(0, T; L^6(\Omega))$ , the convective term  $\varrho u \otimes u$  belongs to  $L^1(0, T; L^p(\Omega))$  for some  $p > 1$  if and only if  $\gamma > 3/2$ .

We close this section with the following definition which generalizes Sobolev spaces to domains depending on time:

**Definition 2.** Let  $\Omega(0) \subset \Omega$  be a regular domain and let  $1 \leq p, q \leq \infty$ . We define, for each  $t \geq 0$ ,  $\Omega(t) = X(t, 0, \Omega(0))$ .

We will say that  $u$  defined on  $\Omega$  belongs (respectively) to  $L^p(0, T; L^q(\Omega(t)))$ ,  $L^p(0, T; W^{1,q}(\Omega(t)))$ ,  $L^p(0, T; W^{2,q}(\Omega(t)))$  for  $1 \leq q \leq 6$  or  $L^p(0, T; W^{3,q}(\Omega(t)))$  for  $1 \leq q \leq 2$ , if  $u \circ X$  belongs (respectively) to  $L^p(0, T; L^q(\Omega(0)))$ ,  $L^p(0, T; W^{1,q}(\Omega(0)))$ ,  $L^p(0, T; W^{2,q}(\Omega(0)))$  or  $L^p(0, T; W^{3,q}(\Omega(0)))$ .

**2. Variational formulation and main result**

We introduce the variational formulation of our problem. Let  $\mathcal{V}$  be the test function space:

$$\mathcal{V} = \{v \in C^\infty((0, T) \times \Omega)^3 \mid v(T) = 0, v(t, \cdot) \in H_0^1(\Omega)^3, \forall t \in [0, T]\}. \quad (2.1)$$

**Definition 3.** We will say that  $(X_S, \varrho, u)$  is a weak solution of the problem (1.1) to (1.9) if:

- (i)  $X_S \in H^1(0, T; H^3(\Omega_S(0)))^3, \varrho \in L^\infty(0, T, L^\gamma(\Omega)), \varrho \geq 0, u \in L^2(0, T, H_0^1(\Omega))^3,$
- (ii) Eq. (1.3) is satisfied almost everywhere on  $(0, T) \times \Omega_S(0),$
- (iii) the continuity equation (1.9) is satisfied in the sense of renormalized solutions,
- (iv) the following weak formulation holds: for all  $v \in \mathcal{V},$

$$\begin{aligned} & \int_0^T \int_\Omega \varrho u \cdot \partial_t v \, dx \, dt + \int_0^T \int_\Omega \varrho (u \otimes u) : \nabla v \, dx \, dt \\ & - \int_0^T \int_{\Omega_S(t)} \sigma_S : \nabla v - \theta \int_0^T ((u(t), v(t)))_{H^3(\Omega_S(t))} \, dt - \int_0^T \int_{\Omega_F(t)} \mathbb{T} : \nabla v \, dx \, dt \\ & + a \int_0^T \int_{\Omega_F(t)} \varrho_F^\gamma \operatorname{div} v \, dx \, dt = - \int_\Omega \varrho^0 u^0 \cdot v(0, \cdot) \, dy. \end{aligned} \quad (2.2)$$

Now, we give the main result of this paper:

**Theorem 1.** Let  $u^0 \in H_0^1(\Omega)^3, \rho_S^0 \in L^\infty(\Omega)$  and  $\varrho_F^0 \in L^\gamma(\Omega_F(0))$  satisfying:

$$0 < \underline{\varrho}_S \leq \varrho_S^0(x) \leq \bar{\varrho}_S, \forall x \in \Omega_S(0) \quad \text{and} \quad \varrho_F^0(x) \geq 0, \forall x \in \Omega_F(0). \quad (2.3)$$

We suppose that  $d(\partial\Omega_S(0), \partial\Omega) > 0.$  Then there exists  $T^* > 0$  depending only on the data and  $\theta$  such that there exists at least one weak solution of the problem (1.1) to (1.9) in the sense of Definition 3 defined on  $(0, T^*).$  This solution is defined until  $T$  given by:

$$T = \sup\{t > 0 \mid d(t) > 0, g(t) > 0 \text{ and } X_S(t, 0, \cdot) \text{ one-to-one}\}, \quad (2.4)$$

where

$$d(t) = d(\partial\Omega_S(t), \partial\Omega) \quad \text{and} \quad g(t) = \inf_{y \in \Omega_S(0)} |\det \nabla X_S(t, 0, y)|.$$

Furthermore, this solution satisfies the energy estimate (1.11).

**Remark 4.** We notice that

$$d(t) \geq d(0) - \sup_{y_0 \in \Omega_S(0)} |X_S(t, 0, y_0) - y_0|.$$

Thanks to the regularizing term,  $X_S$  belongs to  $H^1(0, T; L^\infty(\Omega_S(0)))$  and is bounded by a constant depending on  $\theta$  and  $E_0$ . Therefore,

$$d(t) \geq d(0) - \sqrt{t} \|X_S\|_{H^1(0, T; L^\infty(\Omega_S(0)))} > 0,$$

for  $t$  small enough. Next, we notice that if:

$$\|\nabla X_S(t, 0, \cdot) - \text{Id}\|_{L^\infty((0, T) \times \Omega_S(0))} \leq e,$$

where  $e$  is small enough,  $X_S(t, 0, \cdot)$  is invertible and the orientation is preserved, i.e.,

$$g(t) = \inf_{y \in \Omega_S(0)} |\det \nabla X_S(t, 0, y)| > 0.$$

This will be satisfied during a small time if, for instance, we control the norm of  $X_S$  in  $H^1(0, T; W^{1, \infty}(\Omega_S(0)))$ . These two remarks justify the necessity of a regularization in  $H^1(0, T; W^{1, \infty}(\Omega_S(0)))$ : we want to avoid physical situations which are not consistent (non-preservation of orientation) or which we are not able to work out mathematically (collision between the structure and the boundary or interpenetration).

### 3. Auxiliary results

#### 3.1. Regularity results for a parabolic problem

In this paragraph, we give some regularity results which will be useful later. These results are given in the very special case which interests us.

**Definition 4.** We will say that a bounded domain  $\Omega$  is a set with a  $W^{m, k}$  (resp.  $C^k$ ) boundary if, for each point  $x \in \partial\Omega$ , there exists a neighborhood  $\mathcal{U}$  of  $x$ , a neighborhood  $\mathcal{V}$  of 0 and a  $W^{m, k}$ -diffeomorphism (resp.  $C^k$ -diffeomorphism)  $\Psi : \mathcal{V} \mapsto \mathcal{U}$  such that

$$\Psi(0) = x, \quad \Psi(\Gamma_0(\mathcal{V})) = \partial\Omega \cap \mathcal{U}, \quad \Psi(\mathcal{V}^+) = \Omega \cap \mathcal{U},$$

with:

$$\Gamma_0(\mathcal{V}) = \mathcal{V} \cap \{(x', x_N) \in \mathbb{R}^{N-1} \times \mathbb{R} \mid x_N = 0\},$$

and:

$$\mathcal{V}^+ = \mathcal{V} \cap \{(x', x_N) \in \mathbb{R}^{N-1} \times \mathbb{R} \mid x_N > 0\}.$$



**Proposition 1.** *Let  $\Omega$  be a bounded open set of  $\mathbb{R}^3$  with a  $C^2$  boundary. We consider the following Neumann problem:*

$$\begin{cases} \partial_t w - \operatorname{div}(B\nabla w) + aw + c \cdot \nabla w = f & \text{in } (0, T) \times \Omega, \\ (B\nabla w) \cdot n = 0 & \text{in } (0, T) \times \partial\Omega, \\ w(0) = w_0 & \text{in } \Omega, \end{cases} \quad (3.1)$$

where  $B$  is a symmetric matrix in  $C(0, T; W^{1,6}(\Omega))$  uniformly coercive in space and time.

- (i) *We suppose that  $w_0$  belongs to  $H^2(\Omega)$ ,  $a$  belongs to  $L^2(0, T; C(\Omega))$ ,  $f$  belongs to  $L^2((0, T) \times \Omega)$  and  $c$  belongs to  $C((0, T) \times \Omega)^3$ , then our problem has a unique solution  $w$  in  $L^2(0, T; H^2(\Omega)) \cap H^1(0, T; L^2(\Omega))$  and Eq. (3.1) is satisfied almost everywhere on  $(0, T) \times \Omega$ .*
- (ii) *We suppose that  $w_0$  belongs to  $W^{2,q}(\Omega)$  with  $q = 4/3$ ,  $a$  belongs to  $L^2((0, T) \times \Omega)$ ,  $f$  belongs to  $L^q((0, T) \times \Omega)$  and  $c$  belongs to  $L^2(0, T; L^4(\Omega))^3$ . Moreover, we suppose that our problem has a solution  $w$  in  $L^4(0, T; H^1(\Omega))$ . Then our solution  $w$  belongs in fact to  $W^{1,q}(0, T; L^q(\Omega)) \cap L^q(0, T; W^{2,q}(\Omega))$  and Eq. (3.1) is satisfied almost everywhere on  $(0, T) \times \Omega$ .*

**Proof.** The first result is a classical result of regularity for a parabolic linear equation. The second result derives from a maximal regularity result in  $L^q((0, T) \times \Omega)$  which is given by [17, Chapter IV, Paragraph 9].  $\square$

**Proposition 2.** *Let  $\Omega$  be a bounded open set of  $\mathbb{R}^3$  with a  $C^1$  boundary. We consider the following problem: find  $w$  such that*

$$\begin{aligned} & - \int_0^T \int_{\Omega} w \partial_t \phi + \int_0^T \int_{\Omega} B\nabla w \cdot \nabla \phi + \int_0^T \int_{\Omega} c \cdot \nabla w \phi \\ & = \int_0^T \langle f, \phi \rangle_{H^1(\Omega)' \times H^1(\Omega)} + \int_{\Omega} w_0 \phi(0), \end{aligned} \quad (3.2)$$

holds for each  $\phi \in \overline{\mathcal{D}((0, T) \times \Omega)}$  satisfying  $\phi(T) = 0$ . Here  $B$  is a symmetric matrix in  $H^1(0, T; W^{1,6}(\Omega))$  uniformly coercive in space and time such that  $B(0) = \operatorname{Id}$ . We suppose that  $f$  belongs to  $L^q(0, T; H^1(\Omega)')$  with  $q > 2$ ,  $w_0$  belongs to  $H^1(\Omega)$  and  $c$  belongs to  $L^2(0, T; L^\infty(\Omega))^3$ , then our problem has a unique solution  $w$  in  $L^q(0, \bar{T}; H^1(\Omega))$  where  $\bar{T}$  depends only on the norm of  $B$  in  $H^1(0, T; W^{1,6}(\Omega))$ .

**Proof.** First, we consider that  $B = \operatorname{Id}$  and we show that the following problem has a unique solution: find  $v$  in  $L^q(0, T; H^1(\Omega))$  such that

$$-\int_0^T \int_{\Omega} v \partial_t \phi + \int_0^T \int_{\Omega} \nabla v \cdot \nabla \phi + \int_0^T \int_{\Omega} c \cdot \nabla v \phi = \int_0^T \langle f, \phi \rangle_{H^1(\Omega)' \times H^1(\Omega)} + \int_{\Omega} w_0 \phi(0),$$

holds for each  $\phi \in \mathcal{D}(\overline{(0, T) \times \Omega})$  satisfying  $\phi(T) = 0$ .

We define an intermediary problem: find  $\bar{v}$  in  $L^q(0, T; H^1(\Omega))$  such that for each  $\phi \in \mathcal{D}(\overline{(0, T) \times \Omega})$  satisfying  $\phi(T) = 0$ ,

$$-\int_0^T \int_{\Omega} \bar{v} \partial_t \phi + \int_0^T \int_{\Omega} \nabla \bar{v} \cdot \nabla \phi = \int_0^T \langle f, \phi \rangle_{H^1(\Omega)' \times H^1(\Omega)} + \int_{\Omega} w_0 \phi(0).$$

According to [1], this problem has a unique solution. Next, we define:  $u = v - \bar{v}$ . Then,  $u$  is solution of: for each  $\phi \in \mathcal{D}(\overline{(0, T) \times \Omega})$  such that  $\phi(T) = 0$ ,

$$-\int_0^T \int_{\Omega} u \partial_t \phi + \int_0^T \int_{\Omega} \nabla u \cdot \nabla \phi + \int_0^T \int_{\Omega} c \cdot \nabla u \phi = -\int_0^T \int_{\Omega} c \cdot \nabla \bar{v} \phi. \tag{3.3}$$

If we consider a sequence of functions  $(c_n)$  belonging to  $L^\infty((0, T) \times \Omega)$  which converges to  $c$  in  $L^2(0, T; L^\infty(\Omega))$ , we easily show that the problem (3.3) where we replace  $c$  by  $c_n$  has a unique solution  $u_n$  in  $L^2(0, T; H^2(\Omega)) \cap H^1(0, T; L^2(\Omega))$ . Furthermore, as  $(c_n \cdot \nabla \bar{v})$  is bounded in  $L^r(0, T; L^2(\Omega))$  with  $1 < r < 2$ , we have:

$$\|u_n\|_{L^r(0, T; H^2(\Omega))} + \|u_n\|_{W^{1,r}(0, T; L^2(\Omega))} \leq C \|c_n \cdot \nabla \bar{v}\|_{L^r(0, T; L^2(\Omega))}.$$

From that, we easily deduce that the limit of the sequence  $(u_n)$  is the unique solution of (3.3). Therefore,  $v = u + \bar{v}$  belongs to  $L^r(0, T; H^2(\Omega)) \cap W^{1,r}(0, T; L^2(\Omega))$  and consequently to  $\mathcal{C}(0, T; H^1(\Omega))$ . Thus, in particular,  $v$  belongs to  $L^q(0, T; H^1(\Omega))$ .

To prove that our initial problem (3.1) has a unique solution in  $L^q(0, T; H^1(\Omega))$ , we use a fixed point argument. We consider the application:

$$\begin{aligned} S: L^q(0, T; H^1(\Omega)) &\mapsto L^q(0, T; H^1(\Omega)), \\ \tilde{w} &\mapsto w, \end{aligned} \tag{3.4}$$

where  $w$  is solution of the variational problem: for each  $\phi \in \mathcal{D}(\overline{(0, T) \times \Omega})$  such that  $\phi(T) = 0$ ,

$$\begin{aligned} -\int_0^T \int_{\Omega} w \partial_t \phi + \int_0^T \int_{\Omega} \nabla w \cdot \nabla \phi + \int_0^T \int_{\Omega} c \cdot \nabla w \phi &= \int_0^T \langle f, \phi \rangle_{H^1(\Omega)' \times H^1(\Omega)} + \int_{\Omega} w_0 \phi(0) \\ &+ \int_0^T \int_{\Omega} (\text{Id} - B) \nabla \tilde{w} \cdot \nabla \phi. \end{aligned}$$

According to what precedes, we easily prove that  $S$  is a contraction on an interval  $[0, \bar{T}]$  where  $\bar{T}$  depends only on the norm of  $B$  in  $H^1(0, T; W^{1,6}(\Omega))$ . This implies the existence of a fixed point to  $S$  belonging to  $L^q(0, T; H^1(\Omega))$  which is the solution of (3.2).  $\square$

### 3.2. Regularity results for the Stokes system

This subsection is devoted to an auxiliary regularity result which will be useful in what follows. We prove the existence of a solution to the Stokes problem for a right-hand side belonging to  $L^r$ . Several papers deal with this regularity problem in different classes of domain. In [7], the result is obtained for domains with  $C^2$  boundary and in [3], the case of domains with  $W^{2,\infty}$  boundary is treated. In both papers, the regularity result holds in  $L^r$ , for each  $1 < r < \infty$ . In our result, as the domain is less regular, we have to restrict the possible values of  $r$ .

**Lemma 1.** *Let  $\Omega$  be a bounded domain with a  $W^{2,6}$  boundary. Assume  $1 < r \leq 6$ . We consider the following problem:*

$$\begin{cases} -\Delta v + \nabla p = f & \text{in } (0, T) \times \Omega, \\ \operatorname{div} v = g & \text{in } (0, T) \times \Omega, \\ v = v_\Gamma & \text{in } (0, T) \times \partial\Omega. \end{cases} \tag{3.5}$$

If  $f \in L^r(\Omega)$ ,  $g \in W^{1,r}(\Omega)$ ,  $v_\Gamma \in W^{2-\frac{1}{r},r}(\partial\Omega)$ , then there exists a unique solution to (3.5)  $(v, p) \in W^{2,r}(\Omega) \times W^{1,r}(\Omega)/\mathbb{R}$ . Moreover,

$$\|v\|_{W^{2,r}(\Omega)} + \|p\|_{W^{1,r}(\Omega)/\mathbb{R}} \leq C(\|f\|_{L^r(\Omega)} + \|g\|_{W^{1,r}(\Omega)} + \|v_\Gamma\|_{W^{2-1/r,r}(\partial\Omega)}),$$

where  $C$  only depends on  $\Omega$  and on  $r$ .

**Proof.** This result is obtained by adapting the proof presented in [3]. We give a sketch of the proof in this article and we will emphasize on the differences in our context. We consider an arbitrary domain  $\Omega$  with a  $W^{2,6}$  boundary. We follow the proof of [3]: first, we can always suppose that  $v_\Gamma = 0$  by considering  $v - \phi$  instead of  $v$  where  $\phi \in W^{2,r}(\Omega)$  is a lifting of  $v_\Gamma$ . Next, we consider  $k$  open sets  $\mathcal{U}_i$  introduced in Definition 4 such that  $\partial\Omega \subset \bigcup_{1 \leq i \leq k} \mathcal{U}_i$  and we define a family  $\theta_i$  for  $0 \leq i \leq k$  of functions belonging to  $C^\infty(\mathbb{R}^3)$  such that

$$\begin{aligned} 0 \leq \theta_i \leq 1, \quad \sum_{i=0}^k \theta_i &= 1 \quad \text{in } \mathbb{R}^3, \\ \operatorname{supp} \theta_i &\text{ is a compact set, } \quad \operatorname{supp} \theta_i \subset \mathcal{U}_i, \quad \forall 1 \leq i \leq k, \\ \operatorname{supp} \theta_0 &\subset \mathbb{R}^3 \setminus \partial\Omega \quad \text{and} \quad \theta_0|_\Omega \in C_c^\infty(\Omega). \end{aligned}$$

The first step of this paper consists in proving that the result holds for  $r = 2$ . We define  $(v_i, p_i) = (\theta_i v, \theta_i p)$ . Then  $(v_i, p_i)$  is solution on  $\Omega \cap \mathcal{U}_i$  of:

$$\begin{cases} -\Delta v_i + \nabla p_i = \theta_i f - 2\nabla\theta_i \cdot \nabla v - \Delta\theta_i v - p\nabla\theta_i = F_i, \\ \operatorname{div} v_i = \theta_i g - \nabla\theta_i \cdot v = G_i, \end{cases}$$

where  $F_i$  belongs to  $L^2(\Omega \cap \mathcal{U}_i)$ ,  $G_i$  belongs to  $H^1(\Omega \cap \mathcal{U}_i)$  and these functions satisfy,

$$\|F_i\|_{L^2(\Omega \cap \mathcal{U}_i)} \leq C\|f\|_{L^2(\Omega)} \text{ and } \|G_i\|_{H^1(\Omega \cap \mathcal{U}_i)} \leq C\|g\|_{H^1(\Omega)}.$$

For  $i = 0$ , we can consider that the domain is regular. Thus, we have classical estimates (we refer to [7]):

$$\|u_0\|_{H^2(\mathbb{R}^3)} + \|p_0\|_{H^1(\mathbb{R}^3)/\mathbb{R}} \leq C(\|f\|_{L^2(\Omega)} + \|g\|_{H^1(\Omega)}).$$

According to Definition 4, for each  $i$ , there exists a  $W^{2,6}$ -diffeomorphism  $\Psi_i$  associated to  $\mathcal{U}_i$  and  $\mathcal{V}_i^+$ . We define on  $\mathcal{V}_i^+$ :

$$z_i = v_i \circ \Psi \quad \text{and} \quad q_i = p_i \circ \Psi.$$

From now, we omit the index  $i$ .  $(z, q) \in H_0^1(\mathcal{V}^+) \times L^2(\mathcal{V}^+)$  satisfies the following problem:

$$\begin{cases} a(z, w) + b(w, \operatorname{Jac} \Psi q) = \int_{\mathcal{V}^+} \operatorname{Jac} \Psi (F \circ \Psi) w, & \forall w \in H_0^1(\mathcal{V}^+), \\ b(z, \mu) = - \int_{\mathcal{V}^+} G \circ \Psi \mu, & \forall \mu \in L^2(\mathcal{V}^+), \end{cases} \quad (3.6)$$

where

$$a(v, w) = \sum_{i,j=1}^3 \int_{\mathcal{V}^+} \operatorname{Jac} \Psi a_{i,j} \frac{\partial v}{\partial y_i} \frac{\partial w}{\partial y_j} \quad \text{and} \quad b(w, \mu) = - \sum_{i,j=1}^3 \int_{\mathcal{V}^+} m_{i,j} \mu \frac{\partial w_i}{\partial y_j}.$$

For each  $1 \leq i, j \leq 3$ ,  $m_{i,j}$  and  $a_{i,j}$  are defined by:

$$m_{i,j} = \frac{\partial \Psi_j^{-1}}{\partial y_i} \circ \Psi \quad \text{and} \quad a_{i,j} = \sum_{k=1}^3 m_{k,i} m_{k,j}. \quad (3.7)$$

Coefficients  $a_{i,j}$ ,  $m_{i,j}$  and  $\operatorname{Jac} \Psi$  belong to  $W^{1,6}(\mathcal{V})$ .

Now, we consider a sequence  $(\Psi^n)$  in  $W^{2,\infty}$  which converges to  $\Psi$  and we denote  $(z^n, q^n)$  the solution of the problem (3.6) associated to  $\Psi^n$ . The sequence  $(z^n, q^n)$  converges to  $(z, q)$  in  $H_0^1(\mathcal{V}^+) \times L^2(\mathcal{V}^+)$  and, according to [3], we know that  $(z^n, q^n)$  belongs to  $H^2(\mathcal{V}^+) \times H^1(\mathcal{V}^+)$ . Following the same lines as in [3, Section 3], we will show additional estimates in order to be able to pass to the limit in  $n$ . In [3], estimates are obtained thanks to the translation method. We introduce the following difference quotients; for each vector of the canonical basis  $e_k$ , we define:

$$\delta_k^h v(x) = \frac{v(x + he_k) - v(x)}{h}, \quad \forall x \in \mathcal{V}^+, \quad \forall h > 0 \text{ such that } x + he_k \in \mathcal{V}^+.$$

Thanks to a change of variables, we can prove that  $(\delta_k^h z^n, \delta_k^h(\text{Jac } \Psi^n q^n))$  satisfies:

$$\begin{cases} a(\delta_k^h z^n, w) + b(w, \delta_k^h(\text{Jac } \Psi^n q^n)) = \langle T^n, w \rangle, & \forall w \in H_0^1(\mathcal{V}^+), \\ b(\delta_k^h z^n, \mu) = \langle \chi^n, \mu \rangle, & \forall \mu \in L^2(\mathcal{V}^+). \end{cases} \quad (3.8)$$

We do not write explicitly  $T^n$  and  $\chi^n$  but a straightforward calculation shows that

$$\begin{aligned} \|T^n\|_{H^{-1}(\mathcal{V}^+)} &\leq C_0(\|z^n\|_{W^{1,3}(\mathcal{V}^+)} + \|F \circ \Psi^n\|_{L^2(\mathcal{V}^+)} + \|q^n\|_{L^3(\mathcal{V}^+)}), \\ \|\chi^n\|_{L^2(\mathcal{V}^+)} &\leq C_1(\|G \circ \Psi^n\|_{H^1(\mathcal{V}^+)} + \|z^n\|_{W^{1,3}(\mathcal{V}^+)}). \end{aligned}$$

Here, and in what follows, it is important to notice that the constants  $C_i$  only depend on the norm of  $\text{Jac } \Psi^n$ ,  $a_{i,j}^n$  and  $m_{i,j}^n$  in  $W^{1,6}$ . Thus, by interpolation between  $L^2$  and  $L^6$ , we can assert that there exists  $0 < \theta < 1$  such that

$$\begin{aligned} \|T^n\|_{H^{-1}(\mathcal{V}^+)} &\leq C_2(\|z^n\|_{W^{1,6}(\mathcal{V}^+)}^\theta + \|F \circ \Psi^n\|_{L^2(\mathcal{V}^+)} + \|q^n\|_{L^6(\mathcal{V}^+)}^\theta), \\ \|\chi^n\|_{L^2(\mathcal{V}^+)} &\leq C_3(\|G \circ \Psi^n\|_{H^1(\mathcal{V}^+)} + \|z^n\|_{W^{1,6}(\mathcal{V}^+)}^\theta). \end{aligned}$$

Moreover, as  $(\delta_k^h z^n, \delta_k^h(\text{Jac } \Psi^n q^n))$  is the unique solution of the problem (3.8) in  $H_0^1(\mathcal{V}^+) \times L^2(\mathcal{V}^+)/\mathbb{R}$  which satisfies:

$$\|\delta_k^h z^n\|_{H_0^1(\mathcal{V}^+)} + \|\delta_k^h(\text{Jac } \Psi^n q^n)\|_{L^2(\mathcal{V}^+)} \leq C(\|T^n\|_{H^{-1}(\mathcal{V}^+)} + \|\chi^n\|_{L^2(\mathcal{V}^+)}), \quad (3.9)$$

we obtain, for  $k = 1, 2$ ,

$$\begin{aligned} \|\delta_k^h z^n\|_{H_0^1(\mathcal{V}^+)} + \|\delta_k^h(\text{Jac } \Psi^n q^n)\|_{L^2(\mathcal{V}^+)} &\leq C_4(\|F \circ \Psi^n\|_{L^2(\mathcal{V}^+)} + \|G \circ \Psi^n\|_{H^1(\mathcal{V}^+)}) \\ &\quad + C_5(\|z^n\|_{W^{1,6}(\mathcal{V}^+)}^\theta + \|\text{Jac } \Psi^n q^n\|_{L^6(\mathcal{V}^+)}^\theta). \end{aligned}$$

Since  $(z^n, q^n)$  belongs to  $H^2(\mathcal{V}^+) \times H^1(\mathcal{V}^+)$ , we conclude by passing to the limit in  $h$  that

$$\left\| \frac{\partial z^n}{\partial y_k} \right\|_{H^1(\mathcal{V}^+)} + \left\| \frac{\partial(\text{Jac } \Psi^n q^n)}{\partial y_k} \right\|_{L^2(\mathcal{V}^+)} \leq C_6(\|f\|_{L^2(\Omega)} + \|g\|_{H^1(\Omega)}).$$

Now, we notice that

$$\left\| \frac{\partial q^n}{\partial y_k} \right\|_{L^2(\mathcal{V}^+)} \leq C_7 \left\| \frac{\partial(\text{Jac } \Psi^n q^n)}{\partial y_k} \right\|_{L^2(\mathcal{V}^+)} + C_8 \|\text{Jac } \Psi^n\|_{W^{1,6}(\mathcal{V}^+)} \|q^n\|_{L^6(\mathcal{V}^+)}.$$

Thus, we have: for  $k = 1, 2$ ,

$$\left\| \frac{\partial z^n}{\partial y_k} \right\|_{H^1(\mathcal{V}^+)} + \left\| \frac{\partial q^n}{\partial y_k} \right\|_{L^2(\mathcal{V}^+)} \leq C_9(\|\Psi^n\|_{W^{2,6}(\mathcal{V}^+)}) (\|f\|_{L^2(\Omega)} + \|g\|_{H^1(\Omega)}).$$

For the estimate of  $(\frac{\partial z^n}{\partial y_3}, \frac{\partial q^n}{\partial y_3})$  in  $H^1(\mathcal{V}^+) \times L^2(\mathcal{V}^+)$ , we can exactly follow the proof of [3]. Thus, as  $C_9$  only depends on the norm of  $\Psi^n$  in  $W^{2,6}(\mathcal{V}^+)$ , we are able to pass to the limit in  $n$  and to obtain the following estimate on  $(z, q)$ :

$$\|z\|_{H^2(\mathcal{V}^+)} + \|q\|_{H^1(\mathcal{V}^+)} \leq C(\|f\|_{L^2(\Omega)} + \|g\|_{H^{-1}(\Omega)}).$$

Next, by a change of variables, we come back to the functions  $(v, p)$  on the whole domain  $\Omega$  and we obtain:

$$\|v\|_{H^2(\Omega)} + \|p\|_{H^1(\Omega)} \leq C(\|f\|_{L^2(\Omega)} + \|g\|_{H^{-1}(\Omega)}).$$

Thus, we obtain the desired result for  $r = 2$ . For  $1 < r \leq 6$ , we can adapt the end of the proof to this context without any changes.  $\square$

#### 4. A regularized problem

To prove our existence result, we follow the method of the paper [13]: we first consider a problem with regularizing terms in the fluid equations and we prove that this problem admits a weak solution. We regularize the initial problem in two steps. Firstly, we add an artificial viscosity term in the continuity equation satisfied by the fluid density. The global density is defined by:

$$\varrho = \begin{cases} \varrho_S & \text{in } \Omega_S(t), \\ \varrho_F & \text{in } \Omega_F(t). \end{cases} \tag{4.1}$$

And we define  $\varrho_F$  as the solution of:

$$\begin{cases} \partial_t \varrho_F + \operatorname{div}(\varrho_F u) = \varepsilon \Delta \varrho_F & \text{in } \Omega_F(t), \\ \nabla \varrho_F \cdot n = 0 & \text{on } \partial \Omega_F(t), \\ \varrho_F(0, \cdot) = \varrho_F^0 & \text{in } \Omega_F(0), \end{cases} \tag{4.2}$$

where  $\varepsilon > 0$  is small. On the structure domain, we keep the initial equation:

$$\begin{cases} \partial_t \varrho_S + \operatorname{div}(\varrho_S u) = 0 & \text{on } \Omega_S(t), \\ \varrho_S(0, \cdot) = \varrho_S^0 & \text{on } \Omega_S(0). \end{cases} \tag{4.3}$$

We require some regularity on the initial conditions in order to obtain regularity results on the problem (4.2): we consider initial data  $\varrho_F^0 \in H^2(\Omega_F(0))$  and  $\varrho_S^0 \in H^2(\Omega_S(0))$  such that

$$0 < \underline{\varrho} \leq \varrho^0(x) \leq \bar{\varrho}, \quad \forall x \in \Omega. \tag{4.4}$$

With this viscosity term, we do not keep an energy estimate. Therefore, in order to obtain an energy estimate, we consider the following system for modelling the fluid motion:

$$\partial_t(\varrho_F u_F) + \operatorname{div}(\varrho_F u_F \otimes u_F) + \varepsilon \nabla u_F \nabla \varrho_F + \nabla p - \operatorname{div} \mathbb{T} = 0 \quad \text{in } \Omega_F(t). \quad (4.5)$$

We also strengthen the constitutive law:

$$p = a \varrho_F^\gamma + \delta \varrho_F^\beta, \quad (4.6)$$

where  $\delta > 0$  is small and  $\beta > 4$  is sufficiently large. We will first prove the existence of a weak solution to the variational formulation associated to the regularized problem (1.3), (1.4), (4.2), (4.3) and (4.5) completed by the relations (1.7) and (1.8). In Sections 5 and 6, we will come back to the initial problem by passing to the limit first in  $\varepsilon$  and then in  $\delta$ .

As it is often the case in fluid-structure interaction problems, we are not able to solve the regularized problem directly: we use a linearization procedure. We first solve a linearized finite dimensional problem and then, thanks to a fixed point argument, we will obtain an approximate solution in finite dimension  $(X_S^N, \varrho^N, u^N)$  which satisfies an energy inequality. At last, to obtain a solution of the continuous regularized problem, we pass to the limit in  $N$ .

#### 4.1. The linearized finite-dimensional problem

In this subsection, we will prove existence of a solution for a linearized problem in finite dimension for the velocity. In order to keep an energy estimate, we always consider the continuous equations for the flow and the density.

Let  $(\varphi_i)_{i \in \mathbb{N}}$  be an orthogonal basis of  $H^3(\Omega)^3 \cap H_0^1(\Omega)^3$  and an orthonormal basis of  $H_0^1(\Omega)^3$  endowed with the scalar product:

$$(u, v)_{H_0^1(\Omega)^3} = \int_{\Omega} \nabla u : \nabla v \, dx, \quad \forall u, v \in H_0^1(\Omega)^3.$$

Let  $N$  be a positive fixed integer. We define:

$$\tilde{u}^N(t, x) = \sum_{i=1}^N \tilde{\alpha}_i(t) \varphi_i(x),$$

where  $(\tilde{\alpha}_i)_{1 \leq i \leq N}$  belong to  $L^2(0, T)$ . We suppose that

$$\int_0^T \int_{\Omega} |\nabla \tilde{u}^N(t, x)|^2 \, dx \, dt = \int_0^T \sum_{i=1}^N |\tilde{\alpha}_i(t)|^2 \, dt \leq M, \quad (4.7)$$

where  $M$  is a strictly positive real number. As  $\tilde{u}^N$  is regular, we can solve, for each  $y \in \overline{\Omega}$  the differential equation:

$$\begin{cases} \partial_t \tilde{X}^N(t, 0, y) = \tilde{u}^N(t, \tilde{X}^N(t, 0, y)), \\ \tilde{X}^N(0, 0, y) = y. \end{cases}$$

According to the regularity of the solution of a differential equation with respect to the initial conditions, we can assert that  $\tilde{X}^N$  belongs to  $H^1(0, T; C^1(\bar{\Omega}))$ . Moreover,  $t$  being fixed in  $[0, T]$ ,  $\tilde{X}^N(t, 0, \cdot)$  is invertible from  $\bar{\Omega}$  on  $\bar{\Omega}$ . Thanks to this flow, we can define  $\tilde{\Omega}_S^N(t) = \tilde{X}^N(t, 0, \Omega_S(0))$  and  $\tilde{\Omega}_F^N(t) = \tilde{X}^N(t, 0, \Omega_F(0)) = \Omega \setminus \tilde{\Omega}_S^N(t)$ . As the flow is regular and invertible on  $\Omega$ ,

$$\forall t \in [0, T], \quad d(\tilde{\Omega}_S^N(t), \partial\Omega) > 0.$$

Thus the open set  $\tilde{\Omega}_F^N(t)$  has the same regularity than  $\tilde{\Omega}_S^N(t)$ ,  $\tilde{\Omega}_F^N(t)$  has a  $C^1$  boundary. Then, we define  $\tilde{\varrho}_F^N$  and  $\tilde{\varrho}_S^N$  by:

$$\begin{cases} \partial_t \tilde{\varrho}_F^N + \operatorname{div}(\tilde{\varrho}_F^N \tilde{u}^N) = \varepsilon \Delta \tilde{\varrho}_F^N & \text{in } \tilde{\Omega}_F^N(t), \\ \nabla \tilde{\varrho}_F^N \cdot n = 0 & \text{on } \partial\tilde{\Omega}_F^N(t), \\ \tilde{\varrho}_F^N(0, \cdot) = \varrho_F^0 & \text{in } \Omega_F(0), \end{cases} \quad (4.8)$$

and

$$\begin{cases} \partial_t \tilde{\varrho}_S^N + \operatorname{div}(\tilde{\varrho}_S^N \tilde{u}^N) = 0 & \text{in } \tilde{\Omega}_S^N(t), \\ \varrho_S(0, \cdot) = \varrho_S^0 & \text{in } \Omega_S(0). \end{cases} \quad (4.9)$$

This allows to define also a global density  $\tilde{\varrho}^N$ :

$$\tilde{\varrho}^N = \begin{cases} \tilde{\varrho}_S^N & \text{in } \tilde{\Omega}_S^N(t), \\ \tilde{\varrho}_F^N & \text{in } \tilde{\Omega}_F^N(t). \end{cases} \quad (4.10)$$

The densities  $\tilde{\varrho}_F^N$  and  $\tilde{\varrho}_S^N$  are well defined thanks to the following lemma:

**Lemma 2.** *With the previous notations and hypothesis, the problem defined by (4.8)–(4.10) has a unique solution  $\tilde{\varrho}^N$  in  $L^\infty((0, T) \times \Omega)$  satisfying the energy inequality:*

$$\frac{d}{dt} \int_{\Omega} \tilde{\varrho}^N(t, x)^2 dx + 2\varepsilon \int_{\tilde{\Omega}_F^N(t)} |\nabla \tilde{\varrho}^N(t, x)|^2 dx \leq C,$$

where  $C$  is a constant depending on  $\|\operatorname{div} \tilde{u}^N\|_{L^1(0, T; L^\infty(\Omega))}$ . Furthermore, the solution  $\tilde{\varrho}_S^N$  of (4.9) is given explicitly by the following formula:

$$\tilde{\varrho}_S^N(t, x) = \varrho^0(\tilde{X}^N(0, t, x)) \exp\left(-\int_0^t \operatorname{div} \tilde{u}^N(s, \tilde{X}^N(s, t, x)) ds\right). \quad (4.11)$$

At last,  $\tilde{\varrho}^N$  satisfies the inequality:  $\forall t \in [0, T], \forall x \in \Omega$ ,



$$\underline{\varrho} \exp\left(-\int_0^t \|\operatorname{div} \tilde{u}^N(s)\|_{L^\infty(\Omega)} \, ds\right) \leq \tilde{\varrho}^N(t, x) \leq \bar{\varrho} \exp\left(\int_0^t \|\operatorname{div} \tilde{u}^N(s)\|_{L^\infty(\Omega)} \, ds\right), \tag{4.12}$$

where  $\underline{\varrho}$  et  $\bar{\varrho}$  are defined by inequality (4.4).

**Proof.** As  $\tilde{X}^N$  is a function of  $H^1(0, T; H^3(\bar{\Omega}))$  invertible for any fixed  $t \in [0, T]$ , we can bring back equations (4.8) and (4.9) to reference configurations  $\Omega_F(0)$  and  $\Omega_S(0)$ . Let us define first:

$$\bar{\varrho}_F(t, y) = \tilde{\varrho}_F^N(t, \tilde{X}^N(t, 0, y)) \quad \text{for each } (t, y) \in [0, T] \times \Omega_F(0).$$

Then, after a calculation, we obtain that  $\bar{\varrho}_F$  is solution of:

$$\begin{cases} \partial_t \bar{\varrho}_F - \varepsilon \operatorname{div}(B \nabla \bar{\varrho}_F) + \operatorname{div} \tilde{u}^N(t, \tilde{X}^N(t, 0, \cdot)) \bar{\varrho}_F + c \cdot \nabla \bar{\varrho}_F = 0 & \text{in } \Omega_F(0), \\ (B \nabla \bar{\varrho}_F) \cdot N_y = 0 & \text{on } \partial \Omega_F(0), \end{cases}$$

with

$$B(t, y) = \nabla \tilde{X}^N(t, 0, y)^{-1} \nabla \tilde{X}^N(t, 0, y)^{-t}$$

and

$$c(t, y) = -\frac{\varepsilon}{\det \nabla \tilde{X}^N(t, 0, y)} B(t, y) \nabla (\det \nabla \tilde{X}^N(t, 0, y)).$$

Now, we easily check that we can apply the first part of Proposition 1: we conclude that the function  $\bar{\varrho}_F$  belongs to  $L^2(0, T; H^2(\Omega_F(0))) \cap H^1(0, T; L^2(\Omega_F(0)))$ . Thus  $\tilde{\varrho}_F^N$  belongs to  $L^2(0, T; H^2(\tilde{\Omega}_F^N(t))) \cap H^1(0, T; L^2(\tilde{\Omega}_F^N(t)))$  and Eq. (4.8) is satisfied almost everywhere. By a change of variables, we prove that  $\tilde{\varrho}_S^N$  satisfies the formula (4.11). From this, we deduce that  $\tilde{\varrho}_S^N$  belongs also to  $L^2(0, T; H^2(\tilde{\Omega}_S^N(t))) \cap H^1(0, T; L^2(\tilde{\Omega}_S^N(t)))$ .

At last, we want to show inequality (4.12): on the solid part, it comes directly from (4.11). On the fluid part, we use classical methods involved to show maximum principles. We define:

$$f^N(t, x) = \tilde{\varrho}_F^N(t, x) \exp\left(-\int_0^t \|\operatorname{div} \tilde{u}^N(s, \cdot)\|_{L^\infty(\Omega)} \, ds\right).$$

Then  $f^N$  satisfies almost everywhere the equation:

$$\partial_t f^N + \tilde{u}^N \cdot \nabla f^N + (\operatorname{div} \tilde{u}^N - \|\operatorname{div} \tilde{u}^N(s, \cdot)\|_{L^\infty(\Omega)}) f^N = \varepsilon \Delta f^N \quad \text{in } \tilde{\Omega}_F^N(t).$$

Multiplying successively this equation by  $(f^N - \bar{\varrho})^+$  and by  $(f^N - \underline{\varrho})^-$  with

$$u^+ = \max(0, u), \quad u^- = \min(0, u),$$

we obtain then inequality (4.12).  $\square$

We are now able to linearize the global variational formulation derived from the momentum equation. We look for  $(X^N, \varrho^N, u^N)$  solution of the following problem:

(i) For each  $y \in \overline{\Omega}$ ,  $X^N(t, 0, y)$  is solution of:

$$\begin{cases} \partial_t X^N(t, 0, y) = u^N(t, X^N(t, 0, y)), \\ X^N(0, 0, y) = y. \end{cases} \quad (4.13)$$

(ii) The density is defined by:

$$\varrho^N = \begin{cases} \varrho_S^N & \text{in } \Omega_S^N(t), \\ \varrho_F^N & \text{in } \Omega_F^N(t), \end{cases} \quad (4.14)$$

with  $\Omega_S^N(t) = X^N(t, 0, \Omega_S(0))$  and  $\Omega_F^N(t) = X^N(t, 0, \Omega_F(0)) = \Omega \setminus \overline{\Omega_S^N(t)}$ . Densities  $\varrho_F^N$  and  $\varrho_S^N$  satisfy:

$$\begin{cases} \partial_t \varrho_F^N + \operatorname{div}(\varrho_F^N u^N) = \varepsilon \Delta \varrho_F^N & \text{in } \Omega_F^N(t), \\ \nabla \varrho_F^N \cdot n = 0 & \text{on } \partial \Omega_F^N(t), \\ \varrho_F^N(0, \cdot) = \varrho_F^0 & \text{in } \Omega_F(0), \end{cases} \quad (4.15)$$

and

$$\begin{cases} \partial_t \varrho_S^N + \operatorname{div}(\varrho_S^N u^N) = 0 & \text{in } \Omega_S^N(t), \\ \varrho_S^N(0, \cdot) = \varrho_S^0 & \text{in } \Omega_S(0). \end{cases} \quad (4.16)$$

(iii) At last,  $u^N$  is given by:

$$u^N(t, x) = \sum_{i=1}^N \alpha_i(t) \varphi_i(x), \quad \forall x \in \Omega, \quad \forall t \in [0, T],$$

where  $\alpha_i$ ,  $1 \leq i \leq N$ , belongs to  $H^1(0, T)$  and  $u^N$  satisfies the following problem:

for each  $v^N(t, x) = \sum_{i=1}^N \gamma_i(t) \varphi_i(x)$  where  $\gamma_i$ ,  $1 \leq i \leq N$ , belongs to  $L^2(0, T)$ ,

$$\begin{aligned} & \int_0^T \int_{\Omega} \tilde{\varrho}^N \partial_t u^N \cdot v^N \, dx \, dt + \int_0^T \int_{\Omega} \tilde{\varrho}^N ((\tilde{u}^N \cdot \nabla) u^N) \cdot v^N \, dx \, dt \\ & - \varepsilon \int_0^T \int_{\tilde{\Omega}_F^N(t)} (\nabla v^N \cdot \nabla \tilde{\varrho}^N) u^N \, dx \, dt + \int_0^T \int_{\tilde{\Omega}_S^N(t)} \tilde{\sigma}_S^N : \nabla v^N \, dx \, dt \end{aligned}$$

$$\begin{aligned}
 & + \theta \int_0^T \left( (u^N(t, \cdot), v^N(t, \cdot)) \right)_{H^3(\tilde{\Omega}_S^N(t))} dt + \mu_F \int_0^T \int_{\tilde{\Omega}_F^N(t)} \nabla u^N : \nabla v^N dx dt \\
 & + (\lambda_F + \mu_F) \int_0^T \int_{\tilde{\Omega}_F^N(t)} \operatorname{div} u^N \operatorname{div} v^N dx dt - a \int_0^T \int_{\tilde{\Omega}_F^N(t)} (\tilde{q}^N)^\gamma \operatorname{div} v^N dx dt \\
 & - \delta \int_0^T \int_{\tilde{\Omega}_F^N(t)} (\tilde{q}^N)^\beta \operatorname{div} v^N dx dt = 0.
 \end{aligned} \tag{4.17}$$

Here  $\tilde{\sigma}_S^N$  is defined by: for each  $x \in \tilde{\Omega}_S^N(t)$ ,

$$\tilde{\sigma}_S^N(t, x) = \det \nabla \tilde{X}^N(0, t, x) \nabla \tilde{X}^N(0, t, x)^{-1} \hat{\sigma}_S[\tilde{X}^N](t, \tilde{X}^N(0, t, x)) \nabla \tilde{X}^N(0, t, x)^{-t},$$

where  $\hat{\sigma}_S[\tilde{X}^N]$  is given by (1.5).

Moreover, at initial time,  $u^0$  is a function of  $H_0^1(\Omega)^3$  and has the following writing:

$$u^0 = \sum_{i=1}^\infty \alpha_i^0 \varphi_i \quad \text{with} \quad \sum_{i=1}^\infty |\alpha_i^0|^2 < \infty.$$

Therefore, we prescribe the initial condition:

$$u^N(t=0) = u_0^N := \sum_{i=1}^N \alpha_i^0 \varphi_i \quad \text{or equivalently:} \quad \alpha_i(0) = \alpha_i^0, \quad \forall 1 \leq i \leq N. \tag{4.18}$$

Let us prove that this problem has a unique solution  $(X^N, \varrho^N, u^N)$ . From the variational formulation (4.17), we derive a linear ordinary differential system with the unknowns  $\alpha_i$ ,  $1 \leq i \leq N$  of the form:

$$\begin{cases} A^N \frac{dY^N}{dt} = M^N Y^N + F^N, \\ Y^N(0) = Y^0, \end{cases} \tag{4.19}$$

where  $Y^N = {}^t(\alpha_1, \dots, \alpha_N)$  is a  $N$ -dimensional vector and  $Y^0 = {}^t(\alpha_1^0, \dots, \alpha_N^0)$ . Expressing  $A^N$ ,  $M^N$  and  $F^N$  with respect to  $(\tilde{X}^N, \tilde{q}^N, \tilde{u}^N)$  and the elements of basis  $\varphi_i$ , for each  $1 \leq i \leq N$ , we see that  $A^N(t)$ , for any fixed  $t \in [0, T]$ , is a symmetric definite positive matrix. Furthermore, the matrix  $A^N$  and the vector  $F^N$  are continuous on  $[0, T]$  and the matrix  $M^N$  belongs to  $L^2(0, T)$ . So this system has a unique solution  $(\alpha_i)_{1 \leq i \leq N}$  in  $H^1(0, T)$ .

Furthermore, since  $u^N$  belongs to  $H^1(0, T; H^3(\Omega))$ , the differential equation (4.13) has a unique solution for each fixed  $t \in (0, T)$  and  $X^N$  belongs to  $H^1(0, T; H^3(\Omega))$ . At last, by virtue of Lemma 2,  $\varrho^N$  is uniquely defined in  $L^\infty((0, T) \times \Omega)$ . This provides

the existence of  $(X^N, \varrho^N, u^N)$  solution of the approximated linearized problem defined by Eqs. (4.13) to (4.17).

4.2. The nonlinear finite-dimensional problem

Thanks to the previous step, we will prove the existence of a solution of the approximated nonlinear problem. First of all, taking  $v^N = u^N$  in the variational formulation (4.17), we obtain the following energy estimate for the solution  $(X^N, \varrho^N, u^N)$ :

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \int_{\Omega} \tilde{\varrho}^N |u^N|^2 + (\lambda_F + \mu_F) \int_{\tilde{\Omega}_F^N(t)} |\operatorname{div} u_F^N|^2 \\ & + \mu_F \int_{\tilde{\Omega}_F^N(t)} |\nabla u_F^N|^2 + \theta \|u_S^N\|_{H^3(\tilde{\Omega}_S^N(t))}^2 \leq C_1, \end{aligned}$$

where  $C_1$  depends on  $M$  and on  $N$ . This estimate is obtained using inequality (4.7), estimate on the density  $\tilde{\varrho}^N$  (4.12) and the boundedness of  $\tilde{X}^N$  in  $L^\infty(0, T; \mathcal{C}^1(\Omega))$ . From this inequality, we deduce that

$$\int_{\Omega} |\nabla u^N(t, x)|^2 dx \leq C_2,$$

where  $C_2$  depends on  $M, N$  and the data. Therefore, this provides the existence of a time  $T^N$  depending on  $N$  such that

$$\int_0^{T^N} \sum_{i=1}^N |\alpha_i(t)|^2 \leq M. \tag{4.20}$$

We define the space:

$$\mathcal{C} = \left\{ (\alpha_i)_{1 \leq i \leq N} \in L^2(0, T^N)^N \mid \int_0^{T^N} \sum_{i=1}^N |\alpha_i(t)|^2 \leq M \right\}$$

and the map

$$\begin{aligned} K : \mathcal{C} & \mapsto L^2(0, T^N)^N, \\ \tilde{Y}^N = (\tilde{\alpha}_1, \dots, \tilde{\alpha}_N) & \mapsto Y^N = (\alpha_1, \dots, \alpha_N). \end{aligned}$$

The set  $\mathcal{C}$  is convex and closed in  $L^2(0, T^N)^N$  and the map  $K$  is continuous. Moreover, according to (4.20),  $K(\mathcal{C}) \subset \mathcal{C}$ . To show the existence of a fixed point of  $K$ , we have to prove that  $K(\mathcal{C})$  is a relatively compact set in  $L^2(0, T^N)^N$ . As  $(A^N)^{-1}$  is bounded in  $L^\infty(0, T^N)$ ,  $F^N$  is bounded in  $L^\infty(0, T^N)$  and  $M^N$  is bounded in  $L^2(0, T^N)$  uniformly in  $\tilde{Y}^N$  in  $\mathcal{C}$ , we deduce from (4.19) that  $K(\mathcal{C})$  is a bounded subset of  $W^{1,1}(0, T^N)^N$  and

thus is relatively compact in  $L^2(0, T^N)^N$ . Therefore, we can apply the Schauder’s theorem which gives the existence of a fixed point  $u^N$ .

At this step, we only have the existence of a solution of the approximated nonlinear problem on the interval  $[0, T^N]$  with  $T^N$  depending on  $N$ . This solution satisfies the energy estimate:

$$\begin{aligned} & \frac{1}{2} \int_{\Omega} \varrho^N |u^N|^2 dx + \frac{a}{\gamma - 1} \int_{\Omega_F^N(t)} (\varrho_F^N)^\gamma + \frac{\delta}{\beta - 1} \int_{\Omega_F^N(t)} (\varrho_F^N)^\beta + \mu_F \int_0^t \int_{\Omega_F^N(s)} |\nabla u_F^N|^2 \\ & + (\lambda_F + \mu_F) \int_0^t \int_{\Omega_F^N(s)} |\operatorname{div} u_F^N|^2 + \theta \int_0^t ((u_S^N(s), u_S^N(s)))_{H^3(\Omega_S^N(s))} + \mu_S \int_{\Omega_S(0)} |E(X^N)|^2 \\ & + \frac{\lambda_S}{2} \int_{\Omega_S(0)} |\operatorname{tr} E(X^N)|^2 + \varepsilon \int_0^t \int_{\Omega_F^N(s)} (a\gamma(\varrho_F^N)^{\gamma-2} + \delta\beta(\varrho_F^N)^{\beta-2}) |\nabla \varrho^N|^2 = E_0^N, \end{aligned} \quad (4.21)$$

where  $E_0^N$  tends to  $E_0$  when  $N$  goes to infinity.

We have to prove that we can extend this solution until an arbitrary time  $T$ . To do this, we iterate the process of linearization from new reference configurations  $\Omega_S^N(T^N)$  and  $\Omega_F^N(T^N)$  and from new initial conditions  $u^N(T^N)$  and  $\varrho^N(T^N)$ . Thanks to (4.21) and estimate (4.12) satisfied by  $\varrho^N$ , we show that the solution is defined on a time interval of fixed length independent on  $T^N$ . This allows to extend our solution until the arbitrary time  $T$ .

### 4.3. The continuous problem

Let us pass to the limit in  $N$  to obtain a solution of the continuous regularized problem.

#### 4.3.1. Strong convergence of $(X^N)_{N \in \mathbb{N}}$

First, thanks to the regularizing term in  $\theta$  in the structure equation, we easily obtain a strong convergence result for the flow  $(X^N)_{N \in \mathbb{N}}$ .

For  $N$  sufficiently large, we have  $E_0^N \leq 2E_0$ . So, we deduce from estimate (4.21) that

$$\theta \int_0^T \|u_S^N(t, \cdot)\|_{H^3(\Omega_S^N(t))}^2 dt \leq CE_0. \quad (4.22)$$

Thus, the sequence  $(X^N)_{N \in \mathbb{N}}$  is bounded in  $H^1(0, T; H^3(\Omega_S(0)))$  by a constant depending only on  $E_0$  and  $\theta$ . We denote  $X$  the limit of  $(X^N)_{N \in \mathbb{N}}$  in  $H^1(0, T; H^3(\Omega_S(0)))$  and we define, for each  $t$ ,  $\Omega_S(t) = X(t, 0, \Omega_S(0))$  and  $\Omega_F(t) = \Omega \setminus \overline{\Omega_S(t)}$ . The flow  $X$  satisfies equation (1.3) where  $u$  is the weak limit in  $L^2(0, T; H_0^1(\Omega))^3$  of  $(u^N)_{N \in \mathbb{N}}$ . As the embedding of  $H^1(0, T; H^3(\Omega_S(0)))$  in  $\mathcal{C}(0, T; \mathcal{C}^1(\Omega_S(0)))$  is compact,

$$X^N \rightarrow X \quad \text{in } \mathcal{C}(0, T; \mathcal{C}^1(\Omega_S(0))). \tag{4.23}$$

This allows to assert that

$$\chi_{\Omega_S^N(t)} \rightarrow \chi_{\Omega_S(t)} \quad \text{and} \quad \chi_{\Omega_F^N(t)} \rightarrow \chi_{\Omega_F(t)} \quad \text{in } \mathcal{C}(0, T; L^p(\Omega)), \quad \forall 1 \leq p < \infty, \tag{4.24}$$

where  $\chi_A$  denotes the characteristic function associated to the set  $A$ . Now, the limit  $X$  is only defined on  $\Omega_S(0)$ . In all what follows, we want to avoid collisions between  $\Omega_S(t)$  and the boundary of  $\Omega$  and we want  $X$  to be invertible from  $\Omega_S(0)$  onto  $\Omega_S(t)$ . According to the estimates on  $(X^N)_{N \in \mathbb{N}}$ , these two conditions are valid at least up to a time  $T^* > 0$  depending only on  $\theta$  and initial conditions. Indeed, denoting  $d(t)$  the distance between  $\partial\Omega$  and  $\Omega_S(t)$ , we have:

$$d(t) \geq d_0 - \sup_{y \in \Omega_S(0)} \left| \int_0^t \partial_s X(s, 0, y) \, ds \right|.$$

Thus, thanks to (4.22), we have:

$$d(t) \geq d(0) - C_1 \sqrt{t},$$

where  $C_1$  is a constant depending on  $E_0, \theta$  and on the embedding constant of  $H^2(\Omega_S(0))$  in  $L^\infty(\Omega_S(0))$ .

In the proof, we also want to be able to extend  $X(t, 0, \cdot)$  by an invertible function  $Y(t, \cdot)$  in  $H^1(0, T^*; H^3(\Omega))$  such that boundary points are kept invariant by  $Y$ . This will be useful to come back to the reference configuration  $\Omega_F(0)$  for an equation defined on the moving domain. To do this, we introduce a linear continuous operator:

$$\mathcal{P}: H^3(\Omega_S(0)) \mapsto H^3(\Omega) \cap H_0^1(\Omega)$$

and then, for each  $t \in [0, T]$ , we define the function,

$$Y(t, \cdot) = \text{Id} + \mathcal{P}(X(t, 0, \cdot) - \text{Id}) \quad \text{in } \Omega. \tag{4.25}$$

If we have:

$$\|\nabla Y(t, \cdot) - \text{Id}\|_{L^\infty((0, T^*) \times \Omega)} \leq e,$$

where  $e$  is small enough and depends only on  $\Omega$ , then  $Y(t, \cdot)$  is invertible from  $\Omega$  onto  $\Omega$ , for each  $t$  fixed. But, we remark that

$$\begin{aligned} \|Y(t, \cdot) - \text{Id}\|_{L^\infty(0, T; H^3(\Omega))} &\leq C_{\mathcal{P}} \|X(t, 0, \cdot) - \text{Id}\|_{L^\infty(0, T; H^3(\Omega_S(0)))} \\ &\leq C_{\mathcal{P}} \left\| \int_0^t \partial_s X(s, 0, y) \, ds \right\|_{L^\infty(0, T; H^3(\Omega_S(0)))} \leq C_{\mathcal{P}} C_2 \sqrt{T}, \end{aligned}$$

where  $C_2$  depends only on  $E_0$  and  $C_{\mathcal{P}}$  designs the continuity constant of  $\mathcal{P}$ . Thus, for  $\alpha < 1$  fixed, this provides the existence of a time  $T^*$  depending on  $\alpha, d(0), e_0, \theta$  and  $\Omega$  such that

$$d(t) \geq (1 - \alpha)d(0) \quad \text{for each } t \in [0, T^*] \quad \text{and} \quad \|\nabla Y(t, y) - \text{Id}\|_{L^\infty((0, T^*) \times \Omega)} \leq e_0.$$

In particular,  $Y(t, \cdot)$  is invertible from  $\Omega$  on  $\Omega$  and  $X_S(t, 0, \cdot)$  is invertible from  $\Omega_S(0)$  on  $\Omega_S(t)$ , for each  $t \in [0, T^*]$ . We denote  $X_S(0, t, \cdot)$  the inverse of  $X_S(t, 0, \cdot)$ .

From now on, we work on the interval  $[0, T^*]$ ; the last section will be devoted to the extension of the solution beyond  $T^*$ .

4.3.2. Strong convergence of  $(\varrho^N)_{N \in \mathbb{N}}$

**Lemma 3.** We have the following estimates on  $(\varrho^N)_{N \in \mathbb{N}}: \forall N \in \mathbb{N}$ ,

$$\sup_{0 \leq t \leq T^*} \int_{\Omega} |\varrho^N(t, x)|^\beta \leq C, \quad \varepsilon \int_0^{T^*} \int_{\Omega_F^N(t)} |\nabla \varrho^N|^2 \leq C, \quad \int_0^{T^*} \int_{\Omega} |\varrho^N|^{\beta+1} \leq C. \quad (4.26)$$

**Proof.** On the solid part, as  $\varrho_S^N$  is given by (4.11),  $(\varrho^N)_{N \in \mathbb{N}}$  is bounded on  $L^\infty(0, T^*; L^\infty(\Omega_S^N(t)))$ . On the fluid part, the first estimate comes directly from the energy estimate (4.21). The second estimate is obtained by multiplying (4.15) by  $\varrho_F^N$ . Moreover, according to (4.21),  $((\varrho^N)^{\beta/2})_{N \in \mathbb{N}}$  is bounded in  $L^2(0, T^*; H^1(\Omega_F^N(t)))$ . As

$$H^1(\Omega_F^N(t)) \hookrightarrow L^6(\Omega_F^N(t)),$$

with an embedding constant independent of  $N$  and  $t$ , the sequence  $((\varrho^N)^\beta)_{N \in \mathbb{N}}$  is bounded in  $L^1(0, T^*; L^3(\Omega_F^N(t)))$ . Thus, by interpolation, as  $((\varrho^N)^\beta)_{N \in \mathbb{N}}$  is bounded in  $L^\infty(0, T^*; L^1(\Omega_F^N(t)))$ ,  $((\varrho^N)^\beta)_{N \in \mathbb{N}}$  is bounded in  $L^{4/3}(0, T^*; L^2(\Omega_F^N(t)))$ . From this and the fact that  $\beta$  is taken greater than 4, we deduce the last estimate.  $\square$

We denote by  $\varrho$  the weak limit of  $(\varrho^N)_{N \in \mathbb{N}}$  in  $L^\infty(0, T^*; L^\beta(\Omega))$ . On the solid part, as the velocity is regular, we keep an explicit formula on  $\varrho$ :

$$\varrho(t, x) = \varrho^0(X(0, t, x)) \exp\left(-\int_0^t \text{div } u(s, X(s, t, x)) \, ds\right), \quad \forall x \in \Omega_S(t), \quad (4.27)$$

and  $\varrho$  satisfies on  $\Omega_S(t)$  the continuity equation (4.3). To pass to the limit in the fluid part, we need a result of strong convergence on the density. This is given by the following proposition:

**Proposition 3.** The sequence  $(\varrho^N)$  strongly converges to  $\varrho$  in  $L^\beta((0, T^*) \times \Omega)$ .

**Proof.** As  $(u^N)_{N \in \mathbb{N}}$  is bounded in  $L^2(0, T^*; L^6(\Omega))$  and  $(\varrho^N)_{N \in \mathbb{N}}$  is bounded in  $L^\infty(0, T^*; L^\beta(\Omega))$ , the sequence  $(\varrho^N u^N)_{N \in \mathbb{N}}$  is bounded in  $L^2((0, T^*) \times \Omega)$ . Thus, according to estimate (4.26), Eqs. (4.15) and (4.16),  $(\varrho^N)_{N \in \mathbb{N}}$  is bounded in  $H^1(0, T^*; H^{-1}(\Omega))$ . Moreover, thanks to (4.11) and (4.26),  $(\nabla \varrho^N)_{N \in \mathbb{N}}$  is bounded in  $L^2(0, T^*; L^2(\Omega_F^N(t)))$  and in  $L^2(0, T^*; L^2(\Omega_S^N(t)))$ . Thus, by virtue of Lemma 4 which follows, we obtain the strong convergence of  $(\varrho^N)_{N \in \mathbb{N}}$  in  $L^2((0, T^*) \times \Omega)$ . At last, as  $(\varrho^N)_{N \in \mathbb{N}}$  is bounded in  $L^{\beta+1}((0, T^*) \times \Omega)$  according to (4.26), we can assert that  $(\varrho^N)_{N \in \mathbb{N}}$  strongly converges to  $\varrho$  in  $L^\beta((0, T^*) \times \Omega)$ .  $\square$

We give now an adaptation of Aubin’s lemma to moving domains. A proof of Aubin’s lemma is given in [18, Chapter 1, Theorem 5.1]. We can adapt this proof to our context without main difficulties (for a detailed proof, we refer to [6]).

**Lemma 4.** *Let  $(\varrho^N)_{N \in \mathbb{N}}$  be a bounded sequence in  $L^2((0, T^*) \times \Omega)$  such that*

$$\partial_t \varrho^N \rightharpoonup \partial_t \varrho \quad \text{in } L^2(0, T^*; H^{-1}(\Omega))w,$$

*and  $(\nabla \varrho^N)_{N \in \mathbb{N}}$  is bounded in  $L^2(0, T^*; L^2(\Omega_F^N(t)))$  and in  $L^2(0, T^*; L^2(\Omega_S^N(t)))$ , then*

$$\varrho^N \rightarrow \varrho \quad \text{in } L^2((0, T^*) \times \Omega).$$

Proposition 3 allows us to identify the weak limit of  $(\varrho^N u^N)_{N \in \mathbb{N}}$  in  $L^\infty(0, T^*; L^{2\gamma/(\gamma+1)}(\Omega))^3$  as  $\varrho u$ . The weak formulation associated to Eqs. (4.14) to (4.16) is:

$$-\int_0^{T^*} \int_\Omega \varrho^N (\partial_t \psi + u^N \cdot \nabla \psi) + \varepsilon \int_0^{T^*} \int_{\Omega_F^N(t)} \nabla \varrho^N \nabla \psi = 0, \quad \forall \psi \in \mathcal{D}((0, T^*) \times \Omega). \quad (4.28)$$

Therefore, we can now pass to the limit in this formulation:  $\varrho$  is solution of:

$$-\int_0^{T^*} \int_\Omega \varrho (\partial_t \psi + u \cdot \nabla \psi) + \varepsilon \int_0^{T^*} \int_{\Omega_F(t)} \nabla \varrho \nabla \psi = 0, \quad \forall \psi \in \mathcal{D}((0, T^*) \times \Omega). \quad (4.29)$$

This is equivalent to the system of Eqs. (4.1)–(4.3) expressed in the sense of distributions. To complete this subsection, we set a regularity result on the density:

**Lemma 5.** *The sequence  $(\varrho^N)_{N \in \mathbb{N}}$  is bounded in  $W^{1,q}(0, T^*; L^q(\Omega))$  and in  $L^q(0, T^*; W^{2,q}(\Omega_F^N(t))) \cap L^q(0, T^*; W^{2,q}(\Omega_S^N(t)))$  with  $q = 4/3$ . Moreover, the function  $\varrho$  belongs to  $W^{1,q}(0, T^*; L^q(\Omega))$  and to  $L^q(0, T^*; W^{2,q}(\Omega_F(t))) \cap L^q(0, T^*; W^{2,q}(\Omega_S(t)))$  and the system (4.1)–(4.3) is satisfied almost everywhere.*



**Proof.** On the solid part, estimates on the density come directly from (4.11). On the fluid part, as explained in the previous paragraph, we can extend  $X^N$  on  $\Omega$  by an invertible function  $Y^N$  which belongs to  $H^1(0, T; H^3(\Omega))$ . We define:

$$\bar{\varrho}_F^N(t, y) = \varrho_F^N(t, Y^N(t, y)), \quad \forall t \in (0, T^*), \quad \forall y \in \Omega_F(0).$$

Then  $\bar{\varrho}_F^N$  satisfies on  $(0, T) \times \Omega_F(0)$

$$\partial_t \bar{\varrho}_F^N + c^N \cdot \nabla \bar{\varrho}_F^N - \varepsilon \operatorname{div}(B^N \nabla \bar{\varrho}_F^N) = -\operatorname{div}(\varrho_F^N u^N)(t, Y^N(t, \cdot)),$$

where  $c^N$  and  $B^N$  are defined by:

$$c^N = v^N - \frac{\varepsilon}{\det \nabla Y^N(t, \cdot)} B^N \nabla (\det \nabla Y^N(t, \cdot)), \quad B^N(t, y) = \nabla Y^N(t, y)^{-1} \nabla Y^N(t, y)^{-t}.$$

The sequence  $(B^N)_{N \in \mathbb{N}}$  is bounded in  $H^1(0, T^*; W^{1,6}(\Omega))$  and is uniformly coercive in time and space. The sequence  $(c^N)_{N \in \mathbb{N}}$  is bounded in  $L^2(0, T^*; L^\infty(\Omega))$ . Moreover, as  $(\varrho_F^N u^N)_{N \in \mathbb{N}}$  is bounded in  $L^2(0, T^*; L^{6\beta/(\beta+6)}(\Omega)) \cap L^\infty(0, T^*; L^{2\beta/(\beta+1)}(\Omega))$ , by interpolation, we can say that  $(\varrho_F^N u^N)_{N \in \mathbb{N}}$  is bounded in  $L^4(0, T^*; L^2(\Omega))$ . This allows to apply Proposition 2: the sequence  $(\varrho_F^N)_{N \in \mathbb{N}}$  is bounded in  $L^4(0, \bar{T}; H^1(\Omega))$ , where  $\bar{T}$  depends only on the norm of  $B^N$  in  $H^1(0, T^*; W^{1,6}(\Omega))$ . By writing that

$$\operatorname{div}(\varrho_F^N u^N) = u^N \cdot \nabla \varrho_F^N + \operatorname{div}(u^N) \varrho_F^N,$$

we obtain that  $(\operatorname{div}(\varrho_F^N u^N))_{N \in \mathbb{N}}$  is bounded in  $L^q((0, \bar{T}) \times \Omega_F(0))$  with  $q = 4/3$ . Therefore, according to Proposition 1,  $(\bar{\varrho}_F^N)_{N \in \mathbb{N}}$  is bounded in  $W^{1,q}(0, \bar{T}; L^q(\Omega_F(0))) \cap L^q(0, \bar{T}; W^{2,q}(\Omega_F(0)))$ . To obtain these estimates on the whole interval  $[0, T^*]$ , we iterate the same proof with a change of variables in the new reference configuration  $\Omega_F(\bar{T})$ . In a finite number of steps, we reach the time  $T^*$ .

At last, to get estimates on the limit  $\varrho$ , we adapt the previous argument with  $Y$  instead of  $Y^N$ .  $\square$

#### 4.3.3. Strong convergence of $(u^N)_{N \in \mathbb{N}}$

First, we strengthen the weak convergence of  $(\varrho^N u^N)_{N \in \mathbb{N}}$  in

$$L^\infty(0, T^*; L^{2\gamma/(\gamma+1)}(\Omega))^3.$$

We show that

$$\varrho^N u^N \rightarrow \varrho u \quad \text{in } \mathcal{C}(0, T^*; L_w^{2\gamma/(\gamma+1)}(\Omega))^3. \tag{4.30}$$

To prove this result, it is sufficient to show that, for each  $i \in \mathbb{N}$ ,  $(\int_\Omega \varrho^N u^N \varphi_i)_{N \in \mathbb{N}}$  strongly converges in  $\mathcal{C}(0, T^*)$ . Taking  $v = \chi_{[0,t]} \varphi_i$  in the weak formulation satisfied by  $u^N$ , we obtain:

$$\begin{aligned}
 & \int_{\Omega} \varrho^N(t, x) u^N(t, x) \varphi_i - \int_{\Omega} \varrho^0 u_0^N \varphi_i(x) = \int_0^t \int_{\Omega} \varrho^N u^N \otimes u^N : \nabla \varphi_i \\
 & - \varepsilon \int_0^t \int_{\Omega_F^N(s)} (\nabla u_F^N \cdot \nabla \varrho_F^N) \varphi_i + \int_0^t \int_{\Omega_S^N(s)} \sigma_S^N : \nabla \varphi_i + \theta \int_0^t ((u^N(s, \cdot), \varphi_i(\cdot)))_{H^3(\Omega_S^N(s))} \\
 & + (\lambda_F + \mu_F) \int_0^t \int_{\Omega_F^N(s)} \operatorname{div} u^N \operatorname{div} \varphi_i + \mu_F \int_0^t \int_{\Omega_F^N(s)} \nabla u^N : \nabla \varphi_i \\
 & - a \int_0^t \int_{\Omega_F^N(s)} (\varrho_F^N)^\gamma \operatorname{div} \varphi_i - \delta \int_0^t \int_{\Omega_F^N(s)} (\varrho_F^N)^\beta \operatorname{div} \varphi_i. \tag{4.31}
 \end{aligned}$$

Let us estimate:

$$h_1^N(t) = \int_0^t \int_{\Omega} \varrho^N u^N \otimes u^N : \nabla \varphi_i.$$

As  $(\varrho^N u^N)_{N \in \mathbb{N}}$  is bounded in  $L^\infty(0, T^*; L^{2\gamma/(\gamma+1)}(\Omega))^3$  where  $\gamma > 3/2$  and  $(u^N)_{N \in \mathbb{N}}$  is bounded in  $L^2(0, T^*; L^6(\Omega))$ ,  $(\varrho^N u^N \otimes u^N)_{N \in \mathbb{N}}$  is bounded in  $L^2(0, T^*; L^1(\Omega))$ . Supposing that the basis function  $\varphi_i$  belongs to  $C^1(\Omega)$ , we conclude that  $(h_1^N)_{N \in \mathbb{N}}$  is bounded in  $H^1(0, T^*)$  and therefore,  $(h_1^N)_{N \in \mathbb{N}}$  strongly converges in  $C(0, T^*)$ . We define:

$$h_2^N(t) = -\varepsilon \int_0^t \int_{\Omega_F^N(s)} (\nabla u_F^N \cdot \nabla \varrho_F^N) \varphi_i \, dx \, ds.$$

To estimate this term, we notice that  $(\varrho_F^N)_{N \in \mathbb{N}}$  is bounded in  $L^{4/3}(0, T^*; W^{2,4/3}(\Omega_F^N(t))) \cap L^\infty(0, T^*; L^4(\Omega_F^N(t)))$ , according to Lemma 5. Thus, by integrating by parts, we obtain that  $(\chi_{\Omega_F^N(t)} \nabla \varrho_F^N)_{N \in \mathbb{N}}$  is bounded in  $L^{8/3}(0, T^*; L^2(\Omega))$ . As  $(\nabla u_F^N)_{N \in \mathbb{N}}$  is bounded in  $L^2((0, T^*) \times \Omega)$ , this is sufficient to assert that  $(h_2^N)_{N \in \mathbb{N}}$  strongly converges in  $C(0, T^*)$ . Estimates on the other terms of (4.31) are obtained with the same kind of arguments and we obtain (4.30).

From this, we deduce the strong convergence of the sequence  $(\varrho^N u^N)_{N \in \mathbb{N}}$ :

$$\varrho^N u^N \rightarrow \varrho u \quad \text{in } C(0, T^*; H^{-1}(\Omega))^3. \tag{4.32}$$

#### 4.3.4. Passage to the limit in the weak formulation

To pass to the limit in the weak formulation satisfied by  $u^N$ , we use the strong convergence results given by (4.23), Proposition 3 and (4.32). As  $(u^N)_{N \in \mathbb{N}}$  weakly converges to  $u$  in  $L^2(0, T^*; H_0^1(\Omega))$ , (4.32) implies that

$$\varrho^N u^N \otimes u^N \rightharpoonup \varrho u \otimes u \quad \text{in } \mathcal{D}'((0, T^*) \times \Omega).$$

Therefore, the only remaining difficulty lies in the convergence of the following term:

$$-\varepsilon \int_0^{T^*} \int_{\Omega_F^N(t)} (\nabla u_F^N(t, x) \cdot \nabla \varrho_F^N(t, x)) v(t, x) \, dx \, dt. \tag{4.33}$$

We need a strong convergence result on the sequence  $(\chi_{\Omega_F^N(t)} \nabla \varrho_F^N)_{N \in \mathbb{N}}$  in  $L^2((0, T^*) \times \Omega)$ . If we multiply (4.15) by  $\varrho_F^N$  and (4.2) by  $\varrho_F$  and we integrate in space and in time, we obtain:

$$\begin{aligned} 2\varepsilon \left( \int_0^t \int_{\Omega_F(s)} |\nabla \varrho_F|^2 \, dx \, ds - \int_0^t \int_{\Omega_F^N(s)} |\nabla \varrho_F^N|^2 \, dx \, ds \right) &= \int_0^t \int_{\Omega_F^N(s)} \operatorname{div} u^N |\varrho_F^N|^2 \, dx \, ds \\ &- \int_0^t \int_{\Omega_F(s)} \operatorname{div} u |\varrho_F|^2 \, dx \, ds + \int_{\Omega_F^N(t)} |\varrho_F^N(t, x)|^2 \, dx - \int_{\Omega_F(t)} |\varrho_F(t, x)|^2 \, dx. \end{aligned} \tag{4.34}$$

Using Eq. (4.15), we can reinforce the convergence of  $(\varrho_F^N)_{N \in \mathbb{N}}$  by obtaining a strong convergence result in  $C(0, T^*; L^4_w(\Omega))$ . Therefore, we deduce from (4.34) that  $(\chi_{\Omega_F^N(t)} \nabla \varrho_F^N)_{N \in \mathbb{N}}$  converges to  $\chi_{\Omega_F(t)} \nabla \varrho_F$  in  $L^2((0, T^*) \times \Omega)$ . This result allows to pass to the limit in the term (4.33). For each fixed  $\varepsilon > 0$ , we have thus obtained a solution  $(X_\varepsilon, \varrho_\varepsilon, u_\varepsilon)$  satisfying the following properties:

**Proposition 4.** *For each fixed  $\varepsilon > 0$ , there exists a solution  $(X_\varepsilon, \varrho_\varepsilon, u_\varepsilon)$  of the problem (1.3), (4.1)–(4.3) that satisfies the weak formulation: for each  $v \in \mathcal{V}$ ,*

$$\begin{aligned} &\int_0^{T^*} \int_{\Omega} \varrho_\varepsilon(t, x) u_\varepsilon(t, x) \partial_t v(t, x) \, dx \, dt + \int_0^{T^*} \int_{\Omega} \varrho_\varepsilon(t, x) (u_\varepsilon \otimes u_\varepsilon)(t, x) : \nabla v(t, x) \, dx \, dt \\ &- \varepsilon \int_0^{T^*} \int_{\Omega_{F,\varepsilon}(t)} (\nabla u_{F,\varepsilon}(t, x) \cdot \nabla \varrho_{F,\varepsilon}(t, x)) v(t, x) \, dx \, dt - \int_0^{T^*} \int_{\Omega_{S,\varepsilon}(t)} \sigma_{S,\varepsilon} : \nabla v \\ &- \theta \int_0^{T^*} \left( (u_\varepsilon(t, \cdot), v(t, \cdot)) \right)_{H^3(\Omega_{S,\varepsilon}(t))} \, dt - (\lambda_F + \mu_F) \int_0^{T^*} \int_{\Omega_{F,\varepsilon}(t)} \operatorname{div} u_\varepsilon \operatorname{div} v \, dx \, dt \\ &- \mu_F \int_0^{T^*} \int_{\Omega_{F,\varepsilon}(t)} \nabla u_\varepsilon : \nabla v \, dx \, dt + a \int_0^{T^*} \int_{\Omega_{F,\varepsilon}(t)} \varrho_{F,\varepsilon}^\gamma \operatorname{div} v \, dx \, dt \\ &+ \delta \int_0^{T^*} \int_{\Omega_{F,\varepsilon}(t)} \varrho_{F,\varepsilon}^\beta \operatorname{div} v \, dx \, dt = - \int_{\Omega} \varrho^0 u^0 v(0, \cdot) \, dy. \end{aligned} \tag{4.35}$$

At last,

$$\|X_\varepsilon(t, 0, \cdot)\|_{H^1(0, T^*; H^3(\Omega_S(0)))} \leq C \quad \text{and} \quad \varepsilon \|\nabla_{Q_{F,\varepsilon}}\|_{L^2(0, T^*; L^2(\Omega_{F,\varepsilon}(t)))}^2 \leq C. \quad (4.36)$$

### 5. Passage to the limit in $\varepsilon$

This section is devoted to the passage to the limit in  $\varepsilon$ . The main difficulty lies in the identification of the pressure. We need estimates on the fluid density “up to the boundary”.

With the same arguments as in the previous section, we can assert that the sequence  $(X_\varepsilon)$  converges strongly in  $\mathcal{C}(0, T^*; \mathcal{C}^1(\Omega_S(0)))$  to  $X$  which belongs to  $H^1(0, T^*; H^3(\Omega_S(0)))$ . Furthermore  $X$  satisfies Eq. (1.3). We denote by  $Y_\varepsilon$  the extension of  $X_\varepsilon$  to  $\Omega$  defined in Section 4.3.1.

We also keep the expression (4.27) on the solid part for the limit  $\varrho$  of the sequence  $(\varrho_\varepsilon)$  in  $L^\beta((0, T^*) \times \Omega)$ .

#### 5.1. Estimates on the density

To pass to the limit in the variational formulation, we will need extra estimates for the fluid density. In order to obtain this, we first give two results related to Stokes problem.

We define the linear operators  $R_t^\varepsilon$  and  $P_t^\varepsilon$  by  $R_t^\varepsilon(f) = v$  and  $P_t^\varepsilon(f) = p$  where  $(v, p)$  is the unique solution of the following Stokes problem, for each  $t$  and  $\varepsilon$  fixed:

$$\begin{cases} -\Delta v + \nabla p = f & \text{in } \Omega_{F,\varepsilon}(t), \\ \operatorname{div} v = 0 & \text{in } \Omega_{F,\varepsilon}(t), \\ v = 0 & \text{on } \partial\Omega_{F,\varepsilon}(t), \\ \int_{\Omega_F(0)} p \circ Y_\varepsilon(t, \cdot) = 0. \end{cases} \quad (5.1)$$

First, we recall a result given in [16] which gives existence of solution to the Stokes problem for a right-hand side belonging to  $W^{-1,r}$ . The paper shows that this result holds for a domain with a  $C^1$  boundary or for a Lipschitz domain with a Lipschitz constant small enough.

**Lemma 6.**  $P_t^\varepsilon$  is a continuous operator from  $W^{-1,r}(\Omega_{F,\varepsilon}(t))$  in  $L^r(\Omega_{F,\varepsilon}(t))$  for each  $1 < r < \infty$ . Moreover, the continuity of  $P_t^\varepsilon$  is uniform in  $t$  and in  $\varepsilon$ , i.e.,

$$\|P_t^\varepsilon(f)\|_{L^r(\Omega_{F,\varepsilon}(t))} \leq C \|f\|_{W^{-1,r}(\Omega_{F,\varepsilon}(t))},$$

where  $C$  is independent of  $\varepsilon$  and  $t$ .

Furthermore, according to Lemma 1,  $P_t^\varepsilon$  is also a continuous operator from  $L^r(\Omega_{F,\varepsilon}(t))$  in  $W^{1,r}(\Omega_{F,\varepsilon}(t))$  for each  $1 < r \leq 6$ , and

$$\|P_t^\varepsilon(f)\|_{W^{1,r}(\Omega_{F,\varepsilon}(t))} \leq C \|f\|_{L^r(\Omega_{F,\varepsilon}(t))}.$$

We also give a differentiation result with respect to time for a Stokes problem defined on a moving domain. This result can be proved following the method given in [4].

**Lemma 7.** *Let  $f$  belong to  $C^1(0, T^*; L^r(\Omega_{F,\varepsilon}(t)))$ . We have the following result:*

$$P_t^\varepsilon(\partial_t f) = \partial_t P_t^\varepsilon(f) + \frac{1}{\text{vol}(\Omega_F(0))} \left( \int_{\Omega_F(0)} (\nabla P_t^\varepsilon(f) \cdot v_\varepsilon) \circ Y_\varepsilon(t, \cdot) \right) + p, \quad (5.2)$$

where  $v_\varepsilon$  is the Eulerian velocity associated to  $Y_\varepsilon$  and  $p$  is the pressure solution of the Stokes problem,

$$\begin{cases} -\Delta w + \nabla p = 0 & \text{in } \Omega_{F,\varepsilon}(t), \\ \text{div } w = 0 & \text{in } \Omega_{F,\varepsilon}(t), \\ w = (u_\varepsilon \cdot \nabla) R_t^\varepsilon(f) & \text{on } \partial\Omega_{F,\varepsilon}(t), \\ \int_{\Omega_F(0)} p \circ Y_\varepsilon(t, \cdot) = 0. \end{cases} \quad (5.3)$$

We will now prove global estimates on the density “up to the boundary” of the fluid domain thanks to a method introduced by [20]. At this step, we have to solve difficulties due to the moving interface.

**Lemma 8.**

$$\|Q_{F,\varepsilon}\|_{L^{\gamma+1}(0, T^*; L^{\gamma+1}(\Omega_{F,\varepsilon}(t)))} + \|Q_{F,\varepsilon}\|_{L^{\beta+1}(0, T^*; L^{\beta+1}(\Omega_{F,\varepsilon}(t)))} \leq C, \quad (5.4)$$

where  $C$  depends only on  $\delta$  and the data of the problem.

**Proof.** Formally, we define:

$$\begin{aligned} (u_0, p_0) &= (R_t^\varepsilon, P_t^\varepsilon)(-\Delta u_\varepsilon), (u_1, p_1) \\ &= (R_t^\varepsilon, P_t^\varepsilon)(\partial_t(Q_{F,\varepsilon}u_\varepsilon) + \text{div}(Q_{F,\varepsilon}u_\varepsilon \otimes u_\varepsilon) + \varepsilon \nabla u_\varepsilon \nabla Q_{F,\varepsilon}). \end{aligned}$$

We will check during the proof that these functions are well defined. Then, from the weak formulation (4.35), we deduce the following system satisfied in a weak sense for each  $t$  and  $\varepsilon$  fixed:

$$\begin{cases} -\Delta(u_1 + \mu_F u_0) + \nabla(\mu_F p_0 + p_1 + a(Q_{F,\varepsilon})^\gamma + \delta(Q_{F,\varepsilon})^\beta - (\lambda_F + \mu_F) \text{div } u_\varepsilon) = 0 \\ \quad \text{in } \Omega_{F,\varepsilon}(t), \\ \text{div}(u_1 + \mu_F u_0) = 0 \quad \text{in } \Omega_{F,\varepsilon}(t), \\ u_1 + \mu_F u_0 = 0 \quad \text{on } \partial\Omega_{F,\varepsilon}(t). \end{cases}$$

According to the existence and uniqueness of the pressure up to the addition of a constant, we have:

$$\mu_F p_0 + p_1 + a(Q_{F,\varepsilon})^\gamma + \delta(Q_{F,\varepsilon})^\beta - (\lambda_F + \mu_F) \text{div } u_\varepsilon = c_\varepsilon(t) \quad \text{in } \Omega_{F,\varepsilon}(t),$$

where  $c_\varepsilon(t)$  is a constant depending only on the time and is given by,

$$\int_{\Omega_F(0)} (\mu_F p_0 + p_1 + a(Q_{F,\varepsilon})^\gamma + \delta(Q_{F,\varepsilon})^\beta - (\lambda_F + \mu_F) \operatorname{div} u_\varepsilon) \circ Y_\varepsilon(t, y) - c_\varepsilon(t) \, dy = 0. \tag{5.5}$$

Thus, we have:

$$\begin{aligned} & \int_0^{T^*} \int_{\Omega_{F,\varepsilon}(t)} a(Q_{F,\varepsilon})^{\gamma+1} + \delta(Q_{F,\varepsilon})^{\beta+1} \\ &= \int_0^{T^*} \int_{\Omega_{F,\varepsilon}(t)} ((\lambda_F + \mu_F) \operatorname{div} u_\varepsilon + c_\varepsilon(t)) Q_{F,\varepsilon} \\ & \quad - \int_0^{T^*} \int_{\Omega_{F,\varepsilon}(t)} (\mu_F P_t^\varepsilon(-\Delta u_\varepsilon) + P_t^\varepsilon(\partial_t(Q_{F,\varepsilon} u_\varepsilon))) \\ & \quad + P_t^\varepsilon(\operatorname{div}(Q_{F,\varepsilon} u_\varepsilon \otimes u_\varepsilon)) + P_t^\varepsilon(\varepsilon \nabla u_\varepsilon \nabla Q_{F,\varepsilon}) Q_{F,\varepsilon}. \end{aligned} \tag{5.6}$$

Thanks to the energy estimate satisfied by the solution  $(X_\varepsilon, \varrho_\varepsilon, u_\varepsilon)$  and according to the definition of  $c_\varepsilon$ , we easily show that the first integral in the right-hand side of (5.6) is bounded. For the second integral, we use the properties of  $P_t^\varepsilon$  given by Lemmas 1, 6 and 7. The first term of this integral is defined by:

$$I_1(\varepsilon) = \mu_F \int_0^{T^*} \int_{\Omega_{F,\varepsilon}(t)} P_t^\varepsilon(\Delta u_\varepsilon) Q_{F,\varepsilon}.$$

As  $(u_\varepsilon)$  is bounded in  $L^2(0, T^*; H^1(\Omega))$ , according to Lemma 6,  $P_t^\varepsilon(\Delta u_\varepsilon)$  is bounded in  $L^2(0, T^*; L^2(\Omega_{F,\varepsilon}(t)))$ . This allows to conclude that

$$|I_1(\varepsilon)| \leq C \|P_t^\varepsilon(\Delta u_\varepsilon)\|_{L^2(0, T^*; L^2(\Omega_{F,\varepsilon}(t)))} \|Q_{F,\varepsilon}\|_{L^2(0, T^*; L^2(\Omega_{F,\varepsilon}(t)))}.$$

Next, according to Lemma 7, we have:

$$\begin{aligned} I_2(\varepsilon) &= \int_0^{T^*} \int_{\Omega_{F,\varepsilon}(t)} P_t^\varepsilon(\partial_t(Q_{F,\varepsilon} u_\varepsilon)) Q_{F,\varepsilon} \\ &= \int_0^{T^*} \int_{\Omega_{F,\varepsilon}(t)} \left( \partial_t P_t^\varepsilon(Q_{F,\varepsilon} u_\varepsilon) \right. \\ & \quad \left. + \frac{1}{\operatorname{vol}(\Omega_F(0))} \left( \int_{\Omega_F(0)} (\nabla P_t^\varepsilon(Q_{F,\varepsilon} u_\varepsilon) \cdot v_\varepsilon) \circ Y_\varepsilon(t, \cdot) + p_\varepsilon \right) \right) Q_{F,\varepsilon}, \end{aligned}$$

where  $p_\varepsilon$  is defined by the Stokes problem (5.3) where we replace  $f$  by  $Q_{F,\varepsilon}u_\varepsilon$ . First, as  $Q_{F,\varepsilon}$  satisfies (4.2), we notice that

$$\begin{aligned} I_{2,1}(\varepsilon) &= \int_0^{T^*} \int_{\Omega_{F,\varepsilon}(t)} \partial_t P_t^\varepsilon(Q_{F,\varepsilon}u_\varepsilon)Q_{F,\varepsilon} \\ &= \int_{\Omega_{F,\varepsilon}(T^*)} P_{T^*}^\varepsilon(Q_{F,\varepsilon}(T^*)u_\varepsilon(T^*))Q_{F,\varepsilon}(T^*) \\ &\quad - \int_{\Omega_F(0)} P_0^\varepsilon(Q^0u^0)Q_F^0 + \varepsilon \int_0^{T^*} \int_{\Omega_{F,\varepsilon}(t)} \nabla P_t^\varepsilon(Q_{F,\varepsilon}u_\varepsilon) \cdot \nabla Q_{F,\varepsilon} \\ &\quad - \int_0^{T^*} \int_{\Omega_{F,\varepsilon}(t)} Q_{F,\varepsilon} \nabla P_t^\varepsilon(Q_{F,\varepsilon}u_\varepsilon) \cdot u_\varepsilon. \end{aligned}$$

Thus, thanks to Lemma 1, as  $(P_t^\varepsilon(Q_{F,\varepsilon}u_\varepsilon))$  is bounded in  $\mathcal{C}(0, T^*; W^{1,16/9}(\Omega_{F,\varepsilon}(t)))$  and in  $L^2(0, T^*; H^1(\Omega_{F,\varepsilon}(t)))$ ,  $(I_{2,1}(\varepsilon))$  is uniformly bounded in  $\varepsilon$ . To estimate,

$$I_{2,2}(\varepsilon) = \frac{1}{\text{vol}(\Omega_F(0))} \int_0^{T^*} \left( \int_{\Omega_F(0)} (\nabla P_t^\varepsilon(Q_{F,\varepsilon}u_\varepsilon) \cdot v_\varepsilon) \circ Y_\varepsilon(t, \cdot) \right) \int_{\Omega_{F,\varepsilon}(t)} Q_{F,\varepsilon},$$

we use the boundedness of  $(P_t^\varepsilon(Q_{F,\varepsilon}u_\varepsilon))$  in  $L^2(0, T^*; H^1(\Omega_{F,\varepsilon}(t)))$  and the boundedness of  $(v_\varepsilon)$  in  $L^2(0, T^*; L^2(\Omega_{F,\varepsilon}(t)))$ . At last,  $(p_\varepsilon)$  is bounded in  $L^1(0, T^*; L^2(\Omega_{F,\varepsilon}(t)))$ , as  $((u_\varepsilon \cdot \nabla)R_t^\varepsilon(Q_{F,\varepsilon}u_\varepsilon))$  is bounded in  $L^1(0, T^*; H^{1/2}(\partial\Omega_{F,\varepsilon}(t)))$ . From all these results, we deduce that  $(I_2(\varepsilon))$  is uniformly bounded in  $\varepsilon$ . It remains to study:

$$\begin{aligned} I_3(\varepsilon) &= \int_0^{T^*} \int_{\Omega_{F,\varepsilon}(t)} P_t^\varepsilon(\text{div}(Q_{F,\varepsilon}u_\varepsilon \otimes u_\varepsilon))Q_{F,\varepsilon} \quad \text{and} \\ I_4(\varepsilon) &= \int_0^{T^*} \int_{\Omega_{F,\varepsilon}(t)} P_t^\varepsilon(\varepsilon \nabla u_\varepsilon \nabla Q_{F,\varepsilon})Q_{F,\varepsilon}. \end{aligned}$$

As  $(Q_{F,\varepsilon}u_\varepsilon)$  is bounded in  $L^2(0, T^*; L^2(\Omega_{F,\varepsilon}(t)))$  and  $(u_\varepsilon)$  is bounded in  $L^2(0, T^*; L^6(\Omega_{F,\varepsilon}(t)))$ ,  $(Q_{F,\varepsilon}u_\varepsilon \otimes u_\varepsilon)$  is bounded in  $L^1(0, T^*; L^{3/2}(\Omega_{F,\varepsilon}(t)))$ . Thus, thanks to Lemma 1,  $(P_t^\varepsilon(\text{div}(Q_{F,\varepsilon}u_\varepsilon \otimes u_\varepsilon)))$  is bounded in  $L^1(0, T^*; L^{3/2}(\Omega_{F,\varepsilon}(t)))$  and  $(I_3(\varepsilon))$  is bounded.

Moreover, as  $(\varepsilon \nabla u_\varepsilon \nabla Q_{F,\varepsilon})$  is bounded in  $L^1(0, T^*; L^1(\Omega_{F,\varepsilon}(t)))$ , this sequence is bounded in  $L^1(0, T^*; W^{-1,4/3}(\Omega_{F,\varepsilon}(t)))$ . Thus,  $(I_4(\varepsilon))$  is also bounded. This allows to conclude and to obtain inequality (5.4).  $\square$

5.2. Passage to the limit

To pass to the limit in (4.2) when  $\varepsilon$  goes to 0, we need to identify the limit of  $(\varrho_\varepsilon u_\varepsilon)$  in  $L^\infty(0, T^*; L^{2\gamma/(\gamma+1)}(\Omega))$ . First, as  $\varrho_\varepsilon$  satisfies (4.2), we can strengthen the time convergence and prove that

$$\varrho_\varepsilon \rightarrow \varrho \quad \text{in } \mathcal{C}(0, T^*; L^{\gamma}_w(\Omega)).$$

This implies the following strong convergence result:

$$\varrho_\varepsilon \rightarrow \varrho \quad \text{in } \mathcal{C}(0, T^*; H^{-1}(\Omega)).$$

Therefore, as  $(u_\varepsilon)$  is bounded in  $L^2(0, T^*; H^1_0(\Omega))$ , we can assert that  $(\varrho_\varepsilon u_\varepsilon)$  weakly converges to  $\varrho u$  in  $L^\infty(0, T^*; L^{2\gamma/(\gamma+1)}(\Omega))$ . We are then able to pass to the limit in (4.2): the limit  $\varrho$  is solution of (1.9). Moreover, following exactly the arguments in [13], we show that this equation is satisfied almost everywhere and we can use the regularization procedure introduced in [11] to show that  $\varrho$  satisfies this equation in the sense of renormalized solutions.

As in the Section 4.3.3, we can strengthen the convergence of the sequence  $(\varrho_\varepsilon u_\varepsilon)$  and prove that  $(\varrho_\varepsilon u_\varepsilon)$  converges to  $\varrho u$  in  $\mathcal{C}(0, T^*; L^{2\gamma/(\gamma+1)}_w(\Omega))$ . Now, using compactness of the embedding  $L^{2\gamma/(\gamma+1)}(\Omega) \subset H^{-1}(\Omega)$ , we obtain that  $(\varrho_\varepsilon u_\varepsilon)$  strongly converges to  $\varrho u$  in  $\mathcal{C}(0, T^*; H^{-1}(\Omega))$ . This allows to identify the limit of  $(\varrho_\varepsilon u_\varepsilon \otimes u_\varepsilon)$  in  $\mathcal{D}'((0, T^*) \times \Omega)$ .

To be able to pass to the limit in the weak formulation, it remains to identify the limit of the pressure  $(a\varrho^\gamma_{F,\varepsilon} + \delta\varrho^\beta_{F,\varepsilon})$ . Here, although the fluid domain moves, as it is sufficient to obtain local estimates to identify the pressure, we can follow the method introduced by [13] for a compressible fluid with no moving structures inside. Thanks to Lemma 8, we know that this sequence weakly converges in  $L^{(\beta+1)/\beta}((0, T^*) \times \Omega)$  to  $p$ . We define  $\mathcal{R}(z) = az^\gamma + \delta z^\beta$ . Thus, we want to prove that

$$\mathcal{R}(\varrho_{F,\varepsilon}) \rightharpoonup \mathcal{R}(\varrho_F) \quad \text{in } L^{(\beta+1)/\beta}((0, T^*) \times \Omega). \tag{5.7}$$

The first step consists in proving that

**Lemma 9.** For each  $\varphi \in \mathcal{D}(0, T^*; \mathcal{D}(\Omega_F(t)))$ ,

$$\lim_{\varepsilon \rightarrow 0} \int_0^{T^*} \int_{\Omega} \varphi^2 (\mathcal{R}(\varrho_{F,\varepsilon}) - (\lambda_F + 2\mu_F) \operatorname{div} u_\varepsilon) \varrho_{F,\varepsilon} = \int_0^{T^*} \int_{\Omega} \varphi^2 (p - (\lambda_F + 2\mu_F) \operatorname{div} u) \varrho_F.$$

**Proof.** To prove this lemma, we follow the proof of Lemma 3.2 in [13] by considering the following test functions in (4.35):

$$v = \varphi \mathcal{A}_i[\varphi \varrho_{F,\varepsilon}].$$

The definition and properties of operator  $\mathcal{A}$  are given in [13].  $\square$



Let us consider a nondecreasing sequence  $(\varphi_n)$  of nonnegative functions belonging to  $\mathcal{D}(0, T^*; \mathcal{D}(\Omega_F(t)))$  which converges to  $\chi_{\Omega_F(t)}$  in  $L^p((0, T^*) \times \Omega)$  for each  $1 \leq p < \infty$ . We have, according to Lemma 9, for  $m \leq n$ ,

$$\begin{aligned} \limsup_{\varepsilon \rightarrow 0} \int_0^{T^*} \int_{\Omega} \varphi_m^2 \mathcal{R}(\varrho_{F,\varepsilon}) \varrho_{F,\varepsilon} &\leq \int_0^{T^*} \int_{\Omega} \varphi_n^2 (p - (\lambda_F + 2\mu_F) \operatorname{div} u) \varrho_F \\ &+ (\lambda_F + 2\mu_F) \limsup_{\varepsilon \rightarrow 0} \int_0^{T^*} \int_{\Omega} \varphi_n^2 \varrho_{F,\varepsilon} \operatorname{div} u_{\varepsilon} \\ &\leq \int_0^{T^*} \int_{\Omega_F(t)} p \varrho_F + (\lambda_F + 2\mu_F) \left( \limsup_{\varepsilon \rightarrow 0} \int_0^{T^*} \int_{\Omega_{F,\varepsilon}(t)} \varrho_{F,\varepsilon} \operatorname{div} u_{\varepsilon} - \int_0^{T^*} \int_{\Omega_F(t)} \varrho_F \operatorname{div} u \right) + \eta(n), \end{aligned}$$

with  $\lim_{n \rightarrow \infty} \eta(n) = 0$ . According to Remark 2, as  $\varrho$  satisfies Eq. (1.9) in the sense of renormalized solutions, we can take  $b(z) = z \log(z)$  in (1.13) and we obtain:

$$\int_0^{T^*} \int_{\Omega_F(t)} \varrho_F \operatorname{div} u = \int_{\Omega_F(0)} \varrho_F^0 \log(\varrho_F^0) - \int_{\Omega_F(T^*)} \varrho_F(T^*) \log(\varrho_F(T^*)). \tag{5.8}$$

Moreover, according to Lemma 5,  $\varrho_{F,\varepsilon}$  satisfies (4.2) almost everywhere. By multiplying (4.2) by  $b'(\varrho_{F,\varepsilon})$  where  $b$  is convex and of class  $\mathcal{C}^1$ , we have:

$$\begin{aligned} \partial_t b(\varrho_{F,\varepsilon}) + \operatorname{div}(b(\varrho_{F,\varepsilon}) u_{\varepsilon}) + (b'(\varrho_{F,\varepsilon}) \varrho_{F,\varepsilon} - b(\varrho_{F,\varepsilon})) \operatorname{div} u_{\varepsilon} - \varepsilon \Delta b(\varrho_{F,\varepsilon}) &\leq 0 \\ \text{in } \Omega_{F,\varepsilon}(t). \end{aligned}$$

Taking  $b(z) = z \log(z)$ , we obtain:

$$\int_0^{T^*} \int_{\Omega_{F,\varepsilon}(t)} \varrho_{F,\varepsilon} \operatorname{div} u_{\varepsilon} \leq \int_{\Omega_F(0)} \varrho^0 \log(\varrho^0) - \int_{\Omega_{F,\varepsilon}(T^*)} \varrho_{F,\varepsilon}(T^*) \log(\varrho_{F,\varepsilon}(T^*)). \tag{5.9}$$

As  $b$  is convex, this allows to assert that

$$\limsup_{\varepsilon \rightarrow 0} \int_0^{T^*} \int_{\Omega} \varphi_m^2 \mathcal{R}(\varrho_{F,\varepsilon}) \varrho_{F,\varepsilon} \leq \int_0^{T^*} \int_{\Omega_F(t)} p \varrho_F. \tag{5.10}$$

At last, in order to conclude that (5.7) is satisfied, we use a monotony argument. The application  $\mathcal{R}$  is monotone and thus, for each function  $v$  regular enough, we have:

$$\int_0^{T^*} \int_{\Omega_F(t)} \varphi_m^2 (\mathcal{R}(\varrho_{F,\varepsilon}) - \mathcal{R}(v)) (\varrho_{F,\varepsilon} - v) \geq 0.$$

By passing to the limit in  $\varepsilon$ , we deduce thanks to (5.10):

$$\int_0^{T^*} \int_{\Omega_F(t)} p \varrho_F + \int_0^{T^*} \int_{\Omega_F(t)} \varphi_m^2 \mathcal{R}(v) v - \int_0^{T^*} \int_{\Omega_F(t)} \varphi_m^2 (pv + \mathcal{R}(v) \varrho_F) \geq 0,$$

and then, by passing to the limit in  $m$ , we get,

$$\int_0^{T^*} \int_{\Omega_F(t)} (p - \mathcal{R}(v)) (\varrho_F - v) \geq 0.$$

As this inequality is satisfied for each smooth function  $v$ , we have proved (5.7). In order to conclude this section, we resume the properties of our solution:

**Proposition 5.** *For each fixed  $\delta > 0$ , for each initial data  $\varrho_{S,\delta}^0$  in  $H^2(\Omega_S(0))$ ,  $\varrho_{F,\delta}^0$  in  $H^2(\Omega_F(0))$  satisfying (4.4) and  $u^0$  in  $H_0^1(\Omega)^3$ , there exists a solution  $(X_\delta, \varrho_\delta, u_\delta)$  of (1.3), (1.9) which satisfies the weak formulation: for each  $v \in \mathcal{V}$ ,*

$$\begin{aligned} & \int_0^{T^*} \int_{\Omega} \varrho_\delta(t, x) u_\delta(t, x) \partial_t v(t, x) \, dx \, dt + \int_0^{T^*} \int_{\Omega} \varrho_\delta(t, x) (u_\delta \otimes u_\delta)(t, x) : \nabla v(t, x) \, dx \, dt \\ & - \int_0^{T^*} \int_{\Omega_{S,\delta}(t)} \sigma_{S,\delta} : \nabla v - \theta \int_0^{T^*} ((u_\delta(t, \cdot), v(t, \cdot)))_{H^3(\Omega_{S,\delta}(t))} \, dt \\ & - (\lambda_F + \mu_F) \int_0^{T^*} \int_{\Omega_{F,\delta}(t)} \operatorname{div} u_\delta \operatorname{div} v \, dx \, dt - \mu_F \int_0^{T^*} \int_{\Omega_{F,\delta}(t)} \nabla u_\delta : \nabla v \, dx \, dt \\ & + \int_0^{T^*} \int_{\Omega_{F,\delta}(t)} (a \varrho_\delta^\gamma + \delta \varrho_\delta^\beta) \operatorname{div} v \, dx \, dt = - \int_{\Omega} \varrho^0 u^0 v(0, \cdot) \, dy, \end{aligned} \tag{5.11}$$

with,

$$\sigma_{S,\delta}(t, x) = \det \nabla X_\delta(0, t, x) \nabla X_\delta(0, t, x)^{-1} \hat{\sigma}_S[X_\delta](t, X_\delta(0, t, x)) \nabla X_\delta(0, t, x)^{-t}.$$

At last,  $(X_\delta, \varrho_\delta, u_\delta)$  satisfies the following energy estimate:

$$\begin{aligned}
 & \frac{1}{2} \int_{\Omega} \varrho_{\delta}(t) |u_{\delta}(t)|^2 dx + \frac{a}{\gamma - 1} \int_{\Omega_{F,\delta}(t)} \varrho_{F,\delta}(t)^{\gamma} + \frac{\delta}{\beta - 1} \int_{\Omega_{F,\delta}(t)} \varrho_{F,\delta}(t)^{\beta} \\
 & + \mu_F \int_0^t \int_{\Omega_{F,\delta}(s)} |\nabla u_{F,\delta}|^2 + (\lambda_F + \mu_F) \int_0^t \int_{\Omega_{F,\delta}(s)} |\operatorname{div} u_{F,\delta}|^2 \\
 & + \theta \int_0^t \left( (u_{S,\delta}(s), u_{S,\delta}(s)) \right)_{H^3(\Omega_{S,\delta}(s))} + \mu_S \int_{\Omega_S(0)} |E(X_{\delta}(t, 0, y))|^2 dy \\
 & + \frac{\lambda_S}{2} \int_{\Omega_S(0)} |\operatorname{tr} E(X_{\delta}(t, 0, y))|^2 dy \leq E_{0,\delta}.
 \end{aligned} \tag{5.12}$$

### 6. Passage to the limit in $\delta$

It remains to pass to the limit in the regularizing parameter  $\delta$ . First, we weaken the initial conditions on the density. We consider an initial data  $\varrho_F^0$  in  $L^{\gamma}(\Omega_F(0))$  and a sequence  $(\varrho_{F,\delta}^0)$  of functions belonging to  $H^2(\Omega_F(0))$  such that

$$0 < \delta \leq \varrho_{F,\delta}^0 \leq \delta^{-1/\beta} \quad \text{and} \quad \varrho_{F,\delta}^0 \rightarrow \varrho_F^0 \quad \text{in } L^{\gamma}(\Omega_F(0)) \text{ as } \delta \rightarrow 0.$$

For the structure, we also consider an initial data  $\varrho_S^0$  in  $L^{\infty}(\Omega_S(0))$  and a sequence  $(\varrho_{S,\delta}^0)$  in  $H^2(\Omega_S(0))$  which converges to  $\varrho_S^0$  in  $L^{\infty}(\Omega_S(0))$ .

Let us notice that, with this choice of sequence  $(\varrho_{F,\delta}^0)$ , the initial energy estimate  $E_{0,\delta}$  stays bounded as  $\delta$  tends to 0.

As in the previous section, we show complementary estimates on the sequence  $(\varrho_{\delta})$ :

**Lemma 10.**

$$\|\varrho_{F,\delta}\|_{L^{\gamma+\alpha}(0,T^*;L^{\gamma+\alpha}(\Omega_{F,\delta}(t)))} + \delta \|\varrho_{F,\delta}\|_{L^{\beta+\alpha}(0,T^*;L^{\beta+\alpha}(\Omega_{F,\delta}(t)))} \leq C, \tag{6.1}$$

where  $\alpha$  is a strictly positive real number and  $C$  depends only on the initial data.

**Proof.** We use the same technique as in Lemma 8. At this step, we use the fact that  $\gamma > 3/2$ . We have the identity:

$$\mu_F p_0 + p_1 + a(\varrho_{F,\delta})^{\gamma} + \delta(\varrho_{F,\delta})^{\beta} - (\lambda_F + \mu_F) \operatorname{div} u_{\delta} = c_{\delta}(t) \quad \text{in } \Omega_{F,\delta}(t),$$

where  $c_{\delta}$  is given by Eq. (5.5) where we replaced  $\varepsilon$  by  $\delta$ . Now, the trace of the fluid density is no more defined on the boundary of the fluid domain. Therefore, in order to justify the calculations, we consider a sequence  $(\phi_n)$  in  $\mathcal{D}([0, T^*] \times \Omega_{F,\delta}(t))$

which converges to  $\varrho_{F,\delta}^\alpha$ , where  $\alpha > 0$  has to be fixed, in  $L^\infty(0, T^*; L^{\gamma/\alpha}(\Omega_{F,\delta}(t))) \cap L^\infty(0, T^*; L^{\beta/\alpha}(\Omega_{F,\delta}(t)))$ . We have then:

$$\begin{aligned} \int_0^{T^*} \int_{\Omega_{F,\delta}(t)} (a(\varrho_{F,\delta})^\gamma + \delta(\varrho_{F,\delta})^\beta) \phi_n &= \int_0^{T^*} \int_{\Omega_{F,\delta}(t)} ((\lambda_F + \mu_F) \operatorname{div} u_\delta + c_\delta(t)) \phi_n \\ &- \int_0^{T^*} \int_{\Omega_{F,\delta}(t)} (\mu P_t^\delta(-\Delta u_\delta) + P_t^\delta(\partial_t(\varrho_{F,\delta} u_\delta)) + P_t^\delta(\operatorname{div}(\varrho_{F,\delta} u_\delta \otimes u_\delta))) \phi_n. \end{aligned} \quad (6.2)$$

According to the energy estimate (5.12), the sequence  $(\varrho_{F,\delta})$  is only bounded in  $L^\infty(0, T^*; L^\gamma(\Omega))$  with  $\gamma > 3/2$ . Let us define:

$$J_1(\delta) = (\lambda_F + \mu_F) \int_0^{T^*} \int_{\Omega_{F,\delta}(t)} \operatorname{div} u_\delta \phi_n.$$

Then,

$$\begin{aligned} |J_1(\delta)| &\leq C \|\phi_n\|_{L^2(0, T^*; L^2(\Omega_{F,\delta}(t)))} \|\operatorname{div} u_\delta\|_{L^2(0, T^*; L^2(\Omega_{F,\delta}(t)))} \\ &\leq C \|\phi_n\|_{L^2(0, T^*; L^2(\Omega_{F,\delta}(t)))}. \end{aligned}$$

Next, as  $(c_\delta)$  is bounded in  $L^\infty(0, T^*)$ , we have:

$$|J_2(\delta)| = \left| \int_0^{T^*} c_\delta(t) \int_{\Omega_{F,\delta}(t)} \phi_n \right| \leq C \|\phi_n\|_{L^\infty(0, T^*; L^1(\Omega_{F,\delta}(t)))}.$$

For the other terms, we apply the properties of  $P_t^\delta$  derived from Lemmas 1, 6 and 7. Thus, as  $(u_\delta)$  is bounded in  $L^2(0, T^*; H_0^1(\Omega))$ ,

$$|J_3(\delta)| = \mu \left| \int_0^{T^*} \int_{\Omega_{F,\delta}(t)} P_t^\delta(\Delta u_\delta) \phi_n \right| \leq C \|\phi_n\|_{L^2(0, T^*; L^2(\Omega_{F,\delta}(t)))}.$$

For the term:

$$J_4(\delta) = - \int_0^{T^*} \int_{\Omega_{F,\delta}(t)} P_t^\delta(\partial_t(\varrho_{F,\delta} u_\delta)) \phi_n,$$

we follow the technique of the proof of Lemma 8. We obtain that:

$$|J_4(\delta)| \leq C \|\phi_n\|_{L^\infty(0, T^*; L^{p'}(\Omega_{F,\delta}(t)))} + \left| \int_0^{T^*} \int_{\Omega_{F,\delta}(t)} P_t^\delta(\varrho_{F,\delta} u_\delta) (\partial_t \phi_n + \operatorname{div}(\phi_n u_\delta)) \right|,$$

where  $1 < p' < \infty$  is defined by:  $1/p' = 2/3 - 1/\gamma$ . At last, we show that

$$|J_5(\delta)| = \left| - \int_0^{T^*} \int_{\Omega_{F,\delta}(t)} P_t^\delta(\operatorname{div}(\varrho_{F,\delta} u_\delta \otimes u_\delta)) \phi_n \right| \leq C \|\phi\|_{L^\infty(0, T^*; L^{p'}(\Omega_{F,\delta}(t)))}.$$

Assembling all these estimates and taking  $\alpha \leq 2\gamma/3 - 1$ , we obtain, by passing to the limit in  $n$ , that

$$\int_0^{T^*} \int_{\Omega_{F,\delta}(t)} a(\varrho_{F,\delta})^{\gamma+\alpha} + \delta(\varrho_{F,\delta})^{\beta+\alpha} \leq C \|\varrho_{F,\delta}\|_{L^{2\alpha}(0, T^*; L^{2\alpha}(\Omega_{F,\delta}(t)))}^\alpha + C.$$

If we suppose that  $\alpha < \gamma$ , we deduce from this inequality the desired estimate.  $\square$

To pass to the limit in  $\delta$  in the weak formulation (5.11), we follow exactly the arguments developed in Section 5.2. We obtain that  $(\varrho_\delta u_\delta)$  strongly converges to  $\varrho u$  in  $C(0, T^*; H^{-1}(\Omega))$  and that  $(\varrho_\delta u_\delta \otimes u_\delta)$  strongly converges to  $\varrho u \otimes u$  in  $\mathcal{D}'((0, T^*) \times \Omega)$ . This allows to pass in the limit in the continuity equation satisfied by  $\varrho_\delta$ . Therefore, to conclude the passage to the limit, it is sufficient to prove that

$$\overline{\varrho_F^\gamma} = \varrho_F^\gamma,$$

where  $\overline{\varrho_F^\gamma}$  is the weak limit of the sequence  $(\varrho_{F,\delta}^\gamma)$  in  $L^{(\gamma+\alpha)/\gamma}((0, T^*) \times \Omega)$ .

The end of the proof is now very similar to [13]. We give only the main steps of the proof without detailing. For complementary explanations, we refer to [6]. First, we define a family of cut-off functions:

$$T_k(z) = kT\left(\frac{z}{k}\right),$$

where  $T \in C^\infty(\mathbb{R})$  is a concave function such that

$$T(z) = z, \quad \forall z \leq 1 \quad \text{and} \quad T(z) = 2, \quad \forall z \geq 3.$$

Then, exactly as in [13], we show the following convergence result:

**Lemma 11.** *For each  $k \in \mathbb{N}$ , for each  $\varphi \in \mathcal{D}(0, T^*; \mathcal{D}(\Omega_F(t)))$ ,*

$$\begin{aligned} & \lim_{\delta \rightarrow 0} \int_0^{T^*} \int_{\Omega} \varphi^2 (a \varrho_{F,\delta}^\gamma - (\lambda_F + 2\mu_F) \operatorname{div} u_\delta) T_k(\varrho_{F,\delta}) \\ &= \int_0^{T^*} \int_{\Omega} \varphi^2 (a \overline{\varrho_F^\gamma} - (\lambda_F + 2\mu_F) \operatorname{div} u) \overline{T_k(\varrho_F)}. \end{aligned}$$

From this result, we deduce that

$$\limsup_{\delta \rightarrow 0} \|T_k(\varrho_\delta) - T_k(\varrho)\|_{L^{\gamma+1}((0, T^*) \times \Omega)} \leq c, \tag{6.3}$$

where  $c$  does not depend of  $k$ . This estimate on the solid part is obtained thanks to the strong convergence of  $(\varrho_{S,\delta})$  to  $\varrho_S$  in  $L^\gamma((0, T^*) \times \Omega)$ . This inequality allows to prove that  $\varrho$  satisfies the continuity equation in the sense of renormalized solutions and this fact, thanks to a regularization procedure, allows to identify the limit  $\overline{\varrho \log \varrho}$  of  $(\varrho_\delta \log \varrho_\delta)$ :

$$\overline{\varrho \log \varrho}(t) = (\varrho \log \varrho)(t), \quad \forall x \in \Omega_F(t), \quad \forall t \in [0, T^*].$$

This result implies that  $(\varrho_\delta)$  strongly converges to  $\varrho$  in  $L^1((0, T^*) \times \Omega)$  and allows to identify the pressure.

### 7. Conclusion

To conclude, we will prove that we can extend our solution until the time:

$$T_\alpha = \sup\{t > 0 \mid d(t) > \alpha_1, g(t) > \alpha_2, X_S(t, 0, \cdot) \text{ one-to-one}\},$$

with  $\alpha = (\alpha_1, \alpha_2)$  where  $\alpha_1$  and  $\alpha_2$  are two arbitrary small enough positive real numbers. Thanks to the regularity of our solution, this will give the existence of a solution defined on the interval  $[0, T]$  where  $T$  is defined by (2.4). If  $T^* < T_\alpha$ , we have to extend our solution beyond  $T^*$  on a time interval whose length is independent of  $T^*$ . To do this, we iterate the process with the new reference configurations  $\Omega_S(T^*)$  for the solid domain and  $\Omega_F(T^*)$  for the fluid domain. Initial data are now  $\varrho_F(T^*)$  in  $L^\gamma(\Omega_F(T^*))$ ,  $\varrho_S(T^*)$  in  $L^\infty(\Omega_S(T^*))$  and  $u(T^*)$  in  $H_0^1(\Omega)^3$ . As what has been done on the interval  $[0, T^*]$ , we regularize the data  $\varrho_F(T^*)$  and  $\varrho_S(T^*)$  to solve the problem with  $\varepsilon > 0$ . Conditions on the time existence  $T^*$  are discussed in Section 4.3.1. We resume the arguments to obtain new conditions on the new time existence  $T_1$ ; the solution of the finite dimensional problem satisfies:

$$\theta \int_{T^*}^T \|u_S^N(t, \cdot)\|_{H^3(\Omega_S^N(t))}^2 dt \leq 2E(T^*) \leq 2E_0.$$

This estimate implies that  $(X_S^N(t, T^*, \cdot))_{N \in \mathbb{N}}$  is bounded in  $H^3(\Omega_S(T^*))$  by a constant only depending on  $\theta$  and  $E_0$ . Therefore, we have the following estimate on the distance  $d(t)$  between the structure and the boundary of  $\Omega$  at time  $t$ :

$$d(t) \geq d(T^*) - \sup_{y \in \Omega_S(0)} \left| \int_{T^*}^t \partial_s X_S^N(s, T^*, y) ds \right| \geq \alpha_1 - \bar{C}_1 \sqrt{t},$$

where  $\bar{C}_1$  depends only on  $E_0$ ,  $\theta$  and the embedding constant of  $H^2(\Omega_S(T^*)) \subset L^\infty(\Omega_S(T^*))$ . We can easily prove that this embedding constant only depends on  $E_0$ ,  $\theta$  and  $\alpha_2$ . Thus, on an interval of strictly positive length only depending on  $E_0$ ,  $\theta$  and  $\alpha$ , we have:  $d(t) \geq \alpha/2$ . We also want to extend  $X(t, T^*, \cdot)$  by an invertible function  $\bar{Y}$ . We introduce the operator,

$$\begin{aligned} \bar{\mathcal{P}} : H^3(\Omega_S(T^*)) &\mapsto H^3(\Omega) \cap H_0^1(\Omega), \\ f &\mapsto \mathcal{P}(f \circ X_S(T^*, 0, \cdot)), \end{aligned}$$

and we define:

$$\bar{Y}^N(t, \cdot) = \text{Id} + \bar{\mathcal{P}}(X_S^N(t, T^*, \cdot) - \text{Id}) \quad \text{on } \Omega.$$

Then, we can prove that

$$\|\bar{\mathcal{P}}(f)\|_{H^3(\Omega)} \leq C_{\bar{\mathcal{P}}} \|f\|_{H^3(\Omega_S(T^*))},$$

where  $C_{\bar{\mathcal{P}}}$  only depends on  $\alpha_2$ ,  $\theta$  and  $E_0$ . Therefore, we can reiterate the same work from  $T^*$  on an interval of strictly positive length only depending on  $\alpha$ ,  $E_0$ ,  $\theta$ . We just have to take care that our reference configurations only have a  $H^3$  boundary. We need to weaken the hypothesis of regularity in Proposition 1. This proposition must now be valid in the domain  $\Omega_F(T^*) = Y(T^*, \Omega_F(0))$ . By a change of variables, we can come back to the domain  $\Omega_F(0)$  and the Neumann problem that we obtain satisfies the hypothesis of Proposition 1 on the regular domain  $\Omega_F(0)$ . This allows to obtain the same regularity result for the density. By this way, after a finite number of steps, we reach the time  $T_\alpha$  for an arbitrary  $\alpha$  and thus we conclude the proof of Theorem 1.

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