

Existence of Weak Solutions for the Three-Dimensional Motion of an Elastic Structure in an Incompressible Fluid

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Abstract. We study here the three-dimensional motion of an elastic structure immersed in an incompressible viscous fluid. The structure and the fluid are contained in a fixed bounded connected set Ω . We show the existence of a weak solution for regularized elastic deformations as long as elastic deformations are not too important (in order to avoid interpenetration and preserve orientation on the structure) and no collisions between the structure and the boundary occur. As the structure moves freely in the fluid, it seems natural (and it corresponds to many physical applications) to consider that its rigid motion (translation and rotation) may be large.

The existence result presented here has been announced in [4]. Some improvements have been provided on the model: the model considered in [4] is a simplified model where the structure motion is modelled by decoupled and linear equations for the translation, the rotation and the purely elastic displacement. In what follows, we consider on the structure a model which represents the motion of a structure with large rigid displacements and small elastic perturbations. This model, introduced by [15] for a structure alone, leads to coupled and nonlinear equations for the translation, the rotation and the elastic displacement.

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1. Introduction and equation of the motion

On the elastic structure, we have a rigid motion combined with an elastic motion with small deformations. More precisely, the lagrangian flow X_S is defined by

$$X_S(t, 0, y) = a(t) + Q(t)(y - g_0) + Q(t)\xi(t, y), \quad \forall y \in \Omega_S(0), \quad \forall t \in [0, T], \quad (1.1)$$

where $\Omega_S(0)$ is an open regular set which represents the initial domain occupied by the structure, g_0 is the center of mass of the solid at time $t = 0$, a the translation of the structure, $Q \in SO_3(\mathbb{R})$ the rotation of the structure and ξ the elastic deformation of the structure. The vector $X_S(t, 0, y)$ gives the position at time t of

the particle located in y at initial time. We suppose that

$$\int_{\Omega_S(0)} \rho_S^0(y) \xi(t, y) dy = 0, \quad \int_{\Omega_S(0)} \rho_S^0(y) \xi(t, y) \wedge (y - g_0) dy = 0, \quad (1.2)$$

where ρ_S^0 is the density of the solid at time $t = 0$ which satisfies

$$0 < M_1 \leq \rho_S^0 \leq M_2 \text{ on } \Omega_S(0),$$

where M_1 and M_2 are two positive constants. These conditions mean that the elastic motion is orthogonal to the infinitesimal translations and rotations.

We also define the lagrangian velocity U_S by

$$\begin{aligned} U_S(t, y) &= \partial_t X_S(t, 0, y), \quad \forall y \in \Omega_S(0), \forall t \in [0, T] \\ &= \dot{a}(t) + \omega(t) \wedge Q(t)(y - g_0 + \xi(t, y)) + Q(t) \partial_t \xi(t, y), \end{aligned}$$

where the rotation velocity vector ω is defined in \mathbb{R}^3 by

$$\forall t \in [0, T], \forall x \in \mathbb{R}^3, Q'(t)Q(t)^{-1}x = \omega(t) \wedge x.$$

The lagrangian flow defines at each time the structure domain and the fluid domain. Let us denote

$$\Omega_S(t) = X_S(t, 0, \Omega_S(0)) \text{ and } \Omega_F(t) = \Omega \setminus \overline{\Omega_S(t)},$$

which represent respectively the domain occupied by the structure at time t and the domain occupied by the fluid at time t . Moreover, we suppose that the elastic deformations are small enough to get an invertible flow from $\Omega_S(0)$ onto $\Omega_S(t)$ (this hypothesis of smallness will be satisfied by our solution). Thus, we can define the eulerian velocity u_S by

$$u_S(t, x) = \partial_t X_S(t, 0, X_S(0, t, x)), \quad \forall x \in \Omega_S(t), \forall t \in [0, T], \quad (1.3)$$

where $X_S(0, t, \cdot)$ denotes the inverse of $X_S(t, 0, \cdot)$. By this way, u_S can be expressed with respect to a , Q and ξ :

$$\forall t \in [0, T], \forall x \in \Omega_S(t), u_S(t, x) = \dot{a}(t) + \omega(t) \wedge (x - a(t)) + Q(t) \partial_t \xi(t, X_S(0, t, x)).$$

As the flow is invertible, we can also define

$$X_S(t, s, x) = X_S(t, 0, X_S(0, s, x)), \quad \forall x \in \Omega_S(s), \forall t, s \in [0, T].$$

The vector $X_S(t, s, x)$ gives the position at time t of the particle located in x at time s .

On the fluid domain, we have an eulerian point of view: let u_F be the eulerian velocity of the fluid. At last, we denote u the global eulerian velocity on Ω and X the associated lagrangian flow.

The unknowns of our problem are a , Q or ω the corresponding rotation velocity vector, ξ , u_F , ρ_F the density of the fluid and p the pressure of the fluid.

The motion of the fluid is described by the incompressible viscous Navier–Stokes equations and the evolution of the fluid density is governed by the mass conservation law:

$$\rho_F (\partial_t u_F + (u_F \cdot \nabla) u_F) - \operatorname{div} \sigma_F = 0 \text{ on } \Omega_F(t), \quad (1.4)$$

$$\operatorname{div} u_F = 0 \text{ on } \Omega_F(t), \quad (1.5)$$

$$\partial_t \rho_F + \operatorname{div}(\rho_F u_F) = 0 \text{ on } \Omega. \quad (1.6)$$

The tensor σ_F denotes the Cauchy stress tensor in the fluid and is defined by

$$\sigma_F = 2\nu \varepsilon_x(u) - p Id,$$

where $\nu > 0$ is the viscosity of the fluid and $\varepsilon_x(u) = \frac{1}{2}(\nabla_x u + \nabla_x u^t)$ is the symmetric part of the gradient of u .

We introduce the sets

$$\mathcal{E} = \left\{ \psi \in H^3(\Omega_S(0))^3 / \int_{\Omega_S(0)} \rho_S^0 \psi \, dy = 0, \int_{\Omega_S(0)} \rho_S^0 \psi \wedge (y - g_0) \, dy = 0 \right\} \quad (1.7)$$

and

$$\mathcal{Y} = \left\{ \psi \in \mathcal{E} / \forall r \in \mathbb{R}^3, \int_{\Omega_S(0)} \rho_S^0 (r \wedge (y - g_0)) \cdot (r \wedge (y - g_0 + \psi)) > 0 \right\}. \quad (1.8)$$

In our case, in order to avoid instantaneous collisions or interpenetrations, we need a regularization on the equation describing the structure motion: we add a regularizing term in order to get ξ in $W^{1,\infty}(0, T; H^3(\Omega_S(0)))^3$. The motion of the structure is then given by the following weak formulation:

$$\begin{aligned} & \int_0^T \int_{\Omega_S(0)} \rho_S^0(y) \partial_t^2 X_S(t, 0, y) \cdot V(t, y) \, dy \, dt + \epsilon \int_0^T (\partial_t^2 \xi, \partial_t \eta)_{H^3(\Omega_S(0))} \, dt \\ & + \int_0^T \int_{\Omega_S(0)} \Sigma_E^2(\xi) : \varepsilon_y(\partial_t \eta) = \int_0^T \int_{\partial\Omega_S(t)} (\sigma_F n_x) \cdot V(t, X_S(0, t, x)) \, d\gamma_x \, dt, \end{aligned} \quad (1.9)$$

for all V such that, $\forall (t, y) \in [0, T] \times \Omega_S(0)$

$$V(t, y) = \dot{b}(t) + r(t) \wedge (Q(t)(y - g_0 + \xi(t, y))) + Q(t) \partial_t \eta(t, y), \quad (1.10)$$

with $b \in W^{1,\infty}(0, T)^3$, $r \in L^\infty(0, T)^3$ and $\eta \in W^{1,\infty}(0, T; \mathcal{E})$. If $\xi(t)$ belongs to \mathcal{Y} , we notice that the writing of V is unique.

We have denoted Σ_E^2 the second Piola–Kirchhoff tensor in the solid which is given by

$$\Sigma_E^2(\xi) = \lambda \operatorname{tr}(\varepsilon_y(\xi)) Id + 2\mu \varepsilon_y(\xi),$$

with λ, μ the Lamé constants of the elastic media such that $\lambda + 2\mu > 0$. Moreover, the vector n_x is the outward unit normal to $\partial\Omega_S(t)$ at point x and ϵ is a fixed positive real number. This formulation is introduced in [15] and [20] to which we refer for explanations on the derivation of this model. This model corresponds to the motion of a structure with large rigid displacements and small elastic perturbations.

As it is done in these works in two space dimension, we can deduce from (1.9) the separate equations for the translation, rotation and elastic deformation in three

space dimension: taking successively V equal to \dot{b} , $r(t) \wedge (Q(t)(y - g_0 + \xi(t, y)))$ and $Q(t)\partial_t\eta(t, y)$, we obtain the following equations:

$$m\ddot{a} = \int_{\partial\Omega_S(t)} \sigma_F n_x d\gamma_x, \quad (1.11)$$

$$\begin{aligned} \frac{d}{dt}(J\omega) - \int_{\Omega_S(0)} \rho_S^0(Q\partial_t\xi \wedge (\omega \wedge Q(y - g_0 + \xi))) \\ + \int_{\Omega_S(0)} \rho_S^0 Q(y - g_0 + \xi) \wedge (\omega \wedge Q\partial_t\xi) \\ + \int_{\Omega_S(0)} \rho_S^0 Q(\xi \wedge \partial_t^2\xi) = \int_{\partial\Omega_S(t)} (x - a(t)) \wedge (\sigma_F n_x) d\gamma_x, \end{aligned} \quad (1.12)$$

and, for each $\eta \in W^{1,\infty}(0, T; \mathcal{E})$, we have:

$$\begin{aligned} \int_0^T \int_{\Omega_S(0)} \rho_S^0 \partial_t^2 \xi \cdot \partial_t \eta + 2 \int_0^T \int_{\Omega_S(0)} \rho_S^0 (Q^{-1}\omega \wedge \partial_t \xi) \cdot \partial_t \eta \\ + \epsilon \int_0^T (\partial_t^2 \xi, \partial_t \eta)_{H^3(\Omega_S(0))} dt + \int_0^T \int_{\Omega_S(0)} \rho_S^0 (Q^{-1}\dot{\omega} \wedge \xi) \cdot \partial_t \eta \\ + \int_0^T \int_{\Omega_S(0)} \rho_S^0 (Q^{-1}\omega \wedge (Q^{-1}\omega \wedge (y - g_0 + \xi))) \cdot \partial_t \eta \\ + \int_0^T \int_{\Omega_S(0)} \Sigma_E^2(\xi) : \varepsilon_y(\partial_t \eta) = \int_0^T \int_{\partial\Omega_S(t)} (\sigma_F n_x) \cdot (Q(t)\partial_t \eta(t, X_S(0, t, x))). \end{aligned} \quad (1.13)$$

Equations (1.11), (1.12) and (1.13) are equivalent to the global formulation (1.9). This model is valid as long as $\xi(t)$ belongs to \mathcal{Y} . Here m is the mass of the solid. The matrix J is the inertia tensor related to the center of mass. It is defined by: $\forall t \in [0, T], \forall \omega, r \in \mathbb{R}^3$,

$$J(t)\omega \cdot r = \int_{\Omega_S(t)} \rho_S(t, x) (\omega \wedge (x - a(t))) \cdot (r \wedge (x - a(t))) dx.$$

The density of the solid at time t is denoted ρ_S and satisfies

$$\rho_S(t, x) = \det \nabla X_S(0, t, x) \rho_S^0(X_S(0, t, x)), \quad \forall x \in \Omega_S(t), \quad \forall t \in [0, T].$$

Thus equivalently, $\forall t \in [0, T], \forall \omega, r \in \mathbb{R}^3$,

$$J(t)\omega \cdot r = \int_{\Omega_S(0)} \rho_S^0(y) (\omega \wedge Q(t)(y - g_0 + \xi(t, y))) \cdot (r \wedge Q(t)(y - g_0 + \xi(t, y))) dy. \quad (1.14)$$

At last, we assume that the fluid adheres to the external boundary and that the velocity u is continuous on the interface. So, we have the following boundary condition:

$$u_F(t, x) = \dot{a}(t) + \omega(t) \wedge (x - a(t)) + Q(t)\partial_t \xi(t, X_S(0, t, x)), \quad \forall x \in \partial\Omega_S(t), \quad (1.15)$$

$$u_F(t, x) = 0, \quad \forall x \in \partial\Omega. \quad (1.16)$$

The problem is to be completed by the initial conditions:

$$a(0) = g_0, \dot{a}(0) = a^1, \omega(0) = \omega^0, Q(0) = Id, u_F(0, \cdot) = u_F^0 \text{ on } \Omega_F(0), \quad (1.17)$$

$$\xi(0, \cdot) = 0, \partial_t \xi(0, \cdot) = \xi^1 \text{ on } \Omega_S(0) \text{ and } \rho_F(0, \cdot) = \rho_F^0 \chi_{\Omega_F(0)} \text{ on } \Omega, \quad (1.18)$$

where $\xi^1 \in \mathcal{E}$, $\rho_F^0 \in L^\infty(\Omega)$ and $u_F^0 \in L^2(\Omega_F(0))^3$ satisfy compatibility conditions:

$$\int_{\partial\Omega_S(0)} \xi^1(y) \cdot N_y dy = 0, \quad 0 < M_3 \leq \rho_F^0 \leq M_4 \text{ on } \Omega_F(0), \quad \operatorname{div} u_F^0 = 0 \text{ on } \Omega_F(0), \quad (1.19)$$

$$u_F^0 \cdot n = (a^1 + \omega^0 \wedge (y - g_0) + \xi^1) \cdot n \text{ on } \partial\Omega_S(0), \quad u_F^0 \cdot n = 0 \text{ on } \partial\Omega. \quad (1.20)$$

The vector N_y denotes the outward unit normal to $\partial\Omega_S(0)$ at point y . As explained in the following remark, the first condition of (1.19) results from the incompressibility property of the fluid.

Remark 1.1. Due to the continuity condition on the velocities, we have for each $x \in \partial\Omega_S(t)$,

$$u_F(t, x) = \dot{a}(t) + \omega(t) \wedge (x - a(t)) + Q(t) \partial_t \xi(t, X_S(0, t, x)).$$

The incompressibility hypothesis on u_F implies that

$$\int_{\partial\Omega_S(t)} u_F \cdot n_x d\gamma_x = 0 \text{ (since } u_F = 0 \text{ on } \partial\Omega).$$

Thus, this implies a compatibility condition on ξ :

$$\int_{\partial\Omega_S(t)} (Q(t) \partial_t \xi(t, X_S(0, t, x))) \cdot n_x d\gamma_x = 0$$

expressing that the global volume of the solid has to be constant.

The existence of weak solution in the case of rigid structures has been studied by [8], [10], [13], [14], [16], [17] and [26] for incompressible or compressible fluids (this list of references is by no means exhaustive; we also refer to papers quoted therein). In the case of elastic structure, [11] deals with elastic deformations given by a linear combination of a finite number of modes and the model is valid for infinitesimal rigid displacements. Recently, [9] proved a local existence and regularity result for a structure with pure elastic displacements immersed in a fluid. For studies dealing with an incompressible fluid and an elastic plate occupying a part of the fluid domain boundary, we refer to [1] and [6].

Remark 1.2. We could also regularize the elastic displacement by adding a viscosity term in the elasticity equation in order to have $\xi \in H^1(0, T; H^3(\Omega_S(0)))^3$ (for instance, we could replace $\epsilon (\partial_t^2 \xi, \partial_t \eta)_{H^3(\Omega_S(0))}$ by $\epsilon (\partial_t \xi, \partial_t \eta)_{H^3(\Omega_S(0))}$ in (1.9)). With this choice of regularization, the proof is completely similar to what is presented here.

Remark 1.3. In our study, the regularizing term in the structure equation is an abstract term which is necessary to our study. However, it is worth noting that these kinds of stress tensors with high order spatial derivatives are naturally introduced in the theory of multipolar materials. This theory is built on fundamentals of classical continuum thermodynamics. We refer to [23] and [24] for a description of these materials.

2. Auxiliary results

2.1. Regularity result on Stokes problem

Definition 2.1. We will say that a bounded domain Ω is a set with a $W^{m,k}$ boundary if, for each point $x \in \partial\Omega$, there exists a neighbourhood \mathcal{U} of x , a neighbourhood \mathcal{V} of 0 and a $W^{m,k}$ -diffeomorphism $\Psi : \mathcal{V} \mapsto \mathcal{U}$ such that

$$\Psi(0) = x, \Psi(\Gamma_0(\mathcal{V})) = \partial\Omega \cap \mathcal{U}, \Psi(\mathcal{V}^+) = \Omega \cap \mathcal{U},$$

with

$$\Gamma_0(\mathcal{V}) = \mathcal{V} \cap \{(x', x_N) \in \mathbb{R}^{N-1} \times \mathbb{R} / x_N = 0\}$$

and

$$\mathcal{V}^+ = \mathcal{V} \cap \{(x', x_N) \in \mathbb{R}^{N-1} \times \mathbb{R} / x_N > 0\}.$$

We now give a regularity result on Stokes problem. Let us notice that it does not require the classical hypothesis of smoothness (for instance \mathcal{C}^2 regularity).

Proposition 2.2. *Let Ω be a bounded set of \mathbb{R}^3 with a H^3 boundary. Let $f \in L^2(\Omega)^3$, $g \in H^1(\Omega)$ and $u_\Gamma \in W^{\frac{3}{2}}(\partial\Omega)^3$ be given such that*

$$\int_{\partial\Omega} u_\Gamma \cdot n_x d\gamma_x = \int_{\Omega} g dx.$$

Then the problem

$$\begin{cases} -\Delta u + \nabla p = f & \text{on } \Omega, \\ \operatorname{div} u = g & \text{on } \Omega, \\ u = u_\Gamma & \text{on } \partial\Omega, \end{cases} \quad (2.1)$$

has a unique solution $(u, p) \in H^2(\Omega)^3 \times H^1(\Omega)/\mathbb{R}$. Moreover,

$$\|u\|_{H^2(\Omega)} + \|p\|_{H^1(\Omega)/\mathbb{R}} \leq C(\|f\|_{L^2(\Omega)} + \|g\|_{H^1(\Omega)} + \|u_\Gamma\|_{H^{\frac{3}{2}}(\partial\Omega)}), \quad (2.2)$$

where C is a constant depending only on Ω .

Proof. This proposition derives from a result of [2] which is given for domains with $W^{2,\infty}$ regularity. We adapt this result for less regular open sets. For complementary explanations, we refer to [5]. First, we consider an arbitrary domain Ω with

a $W^{2,\infty}$ boundary and we localize the problem. We follow the proof of [2]: we consider k open sets \mathcal{U}_i introduced in Definition 2.1 such that $\partial\Omega \subset \bigcup_{1 \leq i \leq k} \mathcal{U}_i$. Next, we define a family θ_i for $0 \leq i \leq k$ of functions belonging to $C^\infty(\mathbb{R}^3)$ such that

$$\begin{aligned} 0 \leq \theta_i \leq 1, \quad \sum_{i=0}^k \theta_i &= 1 \text{ in } \mathbb{R}^3, \\ \text{supp } \theta_i &\text{ is a compact set, } \text{supp } \theta_i \subset \mathcal{U}_i, \quad \forall 1 \leq i \leq k, \\ \text{supp } \theta_0 &\subset \mathbb{R}^3 \setminus \partial\Omega \text{ and } \theta_0|_{\Omega} \in C_c^\infty(\Omega). \end{aligned}$$

If Ω is a domain with a $W^{2,\infty}$ boundary, we show complementary estimates by resuming the proof of [2]: as (u, p) solution of (2.1) belongs to $H^2(\Omega) \times H^1(\Omega)$, we prove that:

$$\|\theta_i u\|_{H^2(\mathcal{U}_i^+)} + \|\theta_i p\|_{H^1(\mathcal{U}_i^+)} \leq C(\|f\|_{L^2(\Omega)} + \|g\|_{H^{-1}(\Omega)}), \quad (2.3)$$

where C depends only on the norms of the diffeomorphisms Ψ_i in $W^{2,6}(\mathcal{V}_i)$ (so this inequality does not involve norms in $W^{2,\infty}(\mathcal{V}_i)$). This estimate is shown thanks to a change of variable where we express the Stokes system on \mathcal{V}_i and next, we use the translation method. It consists in working with test functions in the weak formulation written on \mathcal{V}_i^+ of the following type :

$$\delta_k^h v(x) = \frac{v(x + h e_k) - v(x)}{h},$$

where the vectors e_k are the element of the canonical basis. As estimate (2.3) needs only the regularity of Ψ_i in $W^{2,6}(\mathcal{V}_i)$ we are then able to weaken the hypothesis of regularity of the domain. \square

2.2. Differentiability of the solution of Stokes problem with respect to the domain

Here, we give a differentiation result with respect to time for a Stokes problem defined on a moving domain. For the general presentation of the method of differentiation, one can see the papers [21] and [22]. The following proposition can be proven by following the arguments of [3].

Proposition 2.3. *Let X_D be a function belonging to $W^{1,\infty}(0, T; H^3(\Omega))$ such that, for all $t \in [0, T]$, $X_D(t, 0, \cdot)$ is a diffeomorphism in $H^3(\Omega)$.*

For each $t \in [0, T]$, we denote $\Omega_S(t) = X_D(t, 0, \Omega_S(0))$. Let $u_S(t)$ be defined on $\Omega_S(t)$ for all $t \in [0, T]$. We assume that, for all $t \in [0, T]$,

$$u_S(t) = u_S(t, \cdot) \in H^1(\Omega_S(t)) \text{ and } \int_{\partial\Omega_S(t)} u_S(t, x) \cdot n_x \, d\gamma(x) = 0.$$

We also assume that the variations in time of u_S are regular enough:

$$\forall t \in [0, T], \partial_t(u_S(t, X_D(t, 0, \cdot))) \in H^1(\Omega_S(0)).$$

We define, for all $t \in [0, T]$, the extension $u_{S,p}(t)$ of $u_S(t)$ to Ω as the solution on $\Omega_F(t) = \Omega \setminus \overline{\Omega_S(t)}$ of the Stokes problem

$$\begin{cases} -\Delta u_{S,p}(t) + \nabla p(t) = 0 & \text{on } \Omega_F(t), \\ \operatorname{div} u_{S,p}(t) = 0 & \text{on } \Omega_F(t), \\ u_{S,p}(t) = 0 & \text{on } \partial\Omega \cap \partial\Omega_F(t), \\ u_{S,p}(t) = u_S(t) & \text{on } \partial\Omega_S(t) \cap \partial\Omega_F(t). \end{cases}$$

Then, the mapping

$$\begin{aligned} (0, T) &\mapsto L^2(\Omega) \\ u_{S,p} : t &\mapsto u_{S,p}(t) \end{aligned}$$

is differentiable and its derivative denoted by $U_{S,p}$ is defined by

$$U_{S,p}(t, x) = \partial_t(u_{S,p}(t, X_D(t, 0, \cdot))) \circ X_D(0, t, x) - (u_D(t, x) \cdot \nabla)u_{S,p}(t, x), \quad (2.4)$$

where u_D is the eulerian velocity associated to X_D . Moreover, $U_{S,p}$ is solution of the following Stokes problem:

$$\begin{cases} -\Delta U_{S,p}(t, x) + \nabla P(t, x) = 0 & \text{on } \Omega_F(t), \\ \operatorname{div} U_{S,p}(t, x) = 0 & \text{on } \Omega_F(t), \\ U_{S,p}(t, x) = 0 & \text{on } \partial\Omega \cap \partial\Omega_F(t), \\ U_{S,p}(t, x) = U_S(t, x) + v(t, x) & \text{on } \partial\Omega_S(t) \cap \partial\Omega_F(t), \end{cases}$$

where U_S is defined for all $t \in [0, T]$ on $\Omega_S(t)$ by (2.4) replacing $u_{S,p}$ by u_S and:

$$v = (u_D \cdot \nabla)(u_{S,p} - u_S) = (u_D \cdot n)(\nabla(u_{S,p} - u_S) \cdot n).$$

2.3. Condition for injectivity

The following lemma gives a sufficient condition for injectivity. One can find the proof of this result in [7].

Lemma 2.4. *Let Ω be a domain in \mathbb{R}^3 . There exists a real $e_0 > 0$ depending only on Ω such that, for all ϕ in $W^{1,\infty}(\Omega)^3$ satisfying:*

$$\|\nabla\phi - Id\|_{L^\infty(\Omega)} \leq e_0,$$

ϕ is injective on Ω .

3. Variational formulation and main result

We look for a solution (u, ρ_F, a, Q, ξ) such that:

$$\bullet u \in L^\infty(0, T; L^2(\Omega))^3 \cap L^2(0, T; H_0^1(\Omega))^3, \rho_F \in L^\infty((0, T) \times \Omega); \quad (3.1)$$

$$\bullet \text{ The flow corresponding to } u \text{ is defined on } \Omega \text{ and satisfies on } \Omega_S(0): \\ X(t, 0, y) = a(t) + Q(t)(y - g_0) + Q(t)\xi(t, y), \text{ with } a \in W^{1, \infty}(0, T)^3, \\ Q \in W^{1, \infty}(0, T; SO_3(\mathbb{R})), \xi \in W^{1, \infty}(0, T; \mathcal{E}); \quad (3.2)$$

$$\bullet \operatorname{div} u = 0 \text{ on } \Omega_F(t); \quad (3.3)$$

\bullet The density ρ_F satisfies the continuity equation:

$$\begin{cases} \partial_t \rho_F + \operatorname{div}(\rho_F u) = 0 & \text{on } \Omega, \\ \rho_F(0) = \rho_F^0 \chi_{\Omega_F(0)} & \text{on } \Omega. \end{cases} \quad (3.4)$$

We introduce now the concept of variational weak solution. Let \mathcal{V} be the space of test functions:

$$\mathcal{V} = \left\{ v \in H^1(0, T; H_0^1(\Omega))^3 / v(T) = 0, \operatorname{div} v(t, \cdot) = 0 \text{ on } \Omega_F(t), \right. \\ \left. v(t, x) = \dot{b}(t) + r(t) \wedge (x - a(t)) + Q(t) \partial_t \eta(t, X_S(0, t, x)) \text{ on } \Omega_S(t), \quad (3.5) \right. \\ \left. \text{with } b \in H^2(0, T)^3, r \in H^1(0, T)^3, \eta \in H^2(0, T; \mathcal{E}) \right\},$$

where \mathcal{E} is defined by (1.7).

Definition 3.1. We will say that (ρ_F, u) is a weak solution of (1.4) to (1.6) and (1.11) to (1.18) if the conditions (3.1)–(3.4) are satisfied and if the following equality holds, $\forall v \in \mathcal{V}$:

$$\begin{aligned} & m \int_0^T \dot{a} \cdot \ddot{b} dt + \int_0^T (J\omega) \cdot \dot{r} dt + \int_0^T \int_{\Omega_S(0)} \rho_S^0 (Q \partial_t \xi \wedge (\omega \wedge Q(y - g_0 + \xi))) \cdot r dy dt \\ & - \int_0^T \int_{\Omega_S(0)} \rho_S^0 Q(y - g_0 + \xi) \wedge (\omega \wedge Q \partial_t \xi) \cdot r dy dt \\ & - \int_0^T \int_{\Omega_S(0)} \rho_S^0 Q (\xi \wedge \partial_t^2 \xi) \cdot r dy dt + \int_0^T \int_{\Omega_S(0)} \rho_S^0 \partial_t \xi \cdot \partial_t^2 \eta dy dt \\ & - 2 \int_0^T \int_{\Omega_S(0)} \rho_S^0 (Q^{-1} \omega \wedge \partial_t \xi) \cdot \partial_t \eta dy dt - \int_0^T \int_{\Omega_S(0)} \rho_S^0 (Q^{-1} \dot{\omega} \wedge \xi) \cdot \partial_t \eta dy dt \\ & - \int_0^T \int_{\Omega_S(0)} \rho_S^0 (Q^{-1} \omega \wedge (Q^{-1} \omega \wedge (y - g_0 + \xi))) \cdot \partial_t \eta dy dt + \epsilon \int_0^T (\partial_t \xi, \partial_t^2 \eta)_{H^3(\Omega_S(0))} dt \\ & - \int_0^T \int_{\Omega_S(0)} \Sigma_E^2(\xi) : \varepsilon_y(\partial_t \eta) dy dt + \int_0^T \int_{\Omega} \rho_F u \cdot \partial_t v dx dt \\ & + \int_0^T \int_{\Omega} (\rho_F u \otimes u) : \nabla v dx dt - 2\nu \int_0^T \int_{\Omega_F(t)} \varepsilon_x(u) : \varepsilon_x(v) dx dt \end{aligned}$$

$$\begin{aligned}
&= -ma^1 \cdot \dot{b}(0) - (J(0)\omega^0) \cdot r(0) - \int_{\Omega_S(0)} \rho_S^0 \xi^1 \cdot \partial_t \eta(0, \cdot) dy - \epsilon (\xi^1, \partial_t \eta(0, \cdot))_{H^3(\Omega_S(0))} \\
&\quad - \int_{\Omega_F(0)} \rho_F^0 u_F^0 \cdot v(0, \cdot) dy.
\end{aligned}$$

Remark 3.2. A classical solution which satisfies system (1.4) to (1.6) and (1.11) to (1.18) is a weak solution in the sense of Definition 3.1. On the other hand, a simple calculation shows that if (ρ_F, u) is a smooth weak solution, then (ρ_F, u) is a classical solution satisfying equations (1.4) to (1.6) and (1.11) to (1.18).

Remark 3.3. We can notice that the space of test functions depends on the solution itself. This is one of the major difficulties of the fluid-structure interaction problems. One usually solves this difficulty using a fixed point argument.

The rest of the paper is devoted to the proof of the following theorem:

Theorem 3.4. *Let $\xi^1 \in \mathcal{E}$, $u_F^0 \in L^2(\Omega_F(0))^3$, $\rho_F^0 \in L^\infty(\Omega)$, $a^1 \in \mathbb{R}^3$ satisfying (1.19)–(1.20). We suppose that $d(\partial\Omega_S(0), \partial\Omega) > 0$. Then there exists $T^* > 0$ depending only on the data and ϵ such that there exists at least one weak solution of (1.4) to (1.6) and (1.11) to (1.18) in the sense of Definition 3.1 defined on $(0, T^*)$. Moreover, this solution is defined until T given by*

$$T = \sup\{t > 0 / \xi(t) \in \mathcal{Y}, d(t) > 0, \gamma(t) > 0 \text{ and } X_S(t, 0, \cdot) \text{ one-to-one}\},$$

where

$$d(t) = d(\partial\Omega_S(t), \partial\Omega) \text{ and } \gamma(t) = \inf_{y \in \Omega_S(0)} |\det \nabla X_S(t, 0, y)|.$$

At last, this solution satisfies the energy estimate

$$\begin{aligned}
&\frac{1}{2} \int_{\Omega_S(0)} \rho_S^0(y) |\partial_t X_S(t, 0, y)|^2 dy + \frac{1}{2} \int_{\Omega_F(t)} \rho_F(t, x) |u(t, x)|^2 dx \\
&\quad + \frac{1}{2} \epsilon \|\partial_t \xi\|_{H^3(\Omega_S(0))}^2 + \frac{1}{2} \int_{\Omega_S(0)} \Sigma_E^2(\xi)(t, y) : \varepsilon_y(\xi)(t, y) dy \quad (3.6) \\
&\quad + 2\nu \int_0^t \int_{\Omega_F(s)} |\varepsilon_x(u_F(s, x))|^2 dx ds \leq E_0,
\end{aligned}$$

where E_0 is the initial energy defined by:

$$\begin{aligned}
E_0 &= \frac{1}{2} \int_{\Omega_S(0)} \rho_S^0 |a^1 + \omega^0 \wedge (y - g_0) + \xi^1|^2 dy + \frac{1}{2} \int_{\Omega_F(0)} \rho_F^0 |u_F^0|^2 dy \\
&\quad + \frac{1}{2} \epsilon \|\xi^1\|_{H^3(\Omega_S(0))}^2.
\end{aligned}$$

Let $d_0 > 0$, $\gamma_0 > 0$ and $\delta > 0$ be small fixed constants. Up to now, we consider a time $T > 0$ such that, for each (a, ω, ξ) satisfying the energy inequality (3.6), we

have:

$$\forall t \in [0, T], d(t) > d_0, \gamma(t) > \gamma_0, \xi(t) \in \mathcal{Y}_\delta, \quad (3.7)$$

where the set \mathcal{Y}_δ is defined for all $\delta > 0$ by:

$$\mathcal{Y}_\delta = \left\{ \psi \in \mathcal{E} / \forall r \in \mathbb{R}^3, \int_{\Omega_S(0)} \rho_S^0(r \wedge (y - g_0)) \cdot (r \wedge (y - g_0 + \psi)) > \delta \|r\|^2 \right\}. \quad (3.8)$$

This time T is greater than a strictly positive constant depending on the data, ϵ , d_0 , γ_0 and δ . This assertion is true thanks to the regularizing term: if (a, ω, ξ) satisfies (3.6), for $t \leq T$ with some $T > 0$, $\xi(t)$ belongs to \mathcal{Y}_δ , and as X_S is bounded in $W^{1,\infty}(0, T; L^2(\Omega_S(0)))^3$, a is bounded in $W^{1,\infty}(0, T)^3$ and ω is bounded in $L^\infty(0, T)^3$. Thus, finally, X_S is bounded in $W^{1,\infty}(0, T; W^{1,\infty}(\Omega_S(0)))^3$. This implies that collisions or solid interpenetration can not occur instantaneously.

4. Representation of velocities

The goal of this section is to represent any u satisfying conditions (3.1) to (3.4) by velocities defined on fixed reference domains. Here, we adapt the method of [11].

Suppose that we have (w_F, a, Q, ξ_1) such that

- $w_F \in Y^0$ with Y^0 defined by:

$$\begin{aligned} Y^0 &= L^\infty(0, T; L^2(\Omega_F(0)))^3 \cap L^2(0, T; V(\Omega_F(0))) \\ &\text{with } V(\Omega_F(0)) = \{w \in H_0^1(\Omega_F(0))^3 / \operatorname{div} w = 0 \text{ on } \Omega_F(0)\}; \end{aligned} \quad (4.1)$$

- $(a, Q, \xi_1) \in Y_\kappa^1$ where κ equals to $(\kappa_1, \kappa_2, \kappa_3)$ and Y_κ^1 is defined by:

$$\begin{aligned} Y_\kappa^1 &= \{(a, Q, \xi_1) \in W^{1,\infty}(0, T)^3 \times W^{1,\infty}(0, T; SO_3(\mathbb{R})) \times W^{1,\infty}(0, T; \mathcal{E}) / \\ &\|a - g_0\|_{L^\infty(0, T)} \leq \kappa_0, \|Q - Id\|_{L^\infty(0, T)} \leq \kappa_1, \|\xi_1\|_{L^\infty(0, T; H^3(\Omega_S(0)))^3} \leq \kappa_2\}. \end{aligned}$$

The positive real numbers κ_0 , κ_1 and κ_2 will be defined later.

Let (a, Q, ξ_1) be given in Y_κ^1 . We can define a flow X_S by (1.1). A priori, this flow is not compatible with an incompressible fluid velocity: indeed, we have shown in Remark 1.1 that the coupling implies a compatibility condition on ξ which is not automatically satisfied by an arbitrary elastic deformation. Therefore, we add in the definition of the flow a term of dilation or compression of the solid volume which will balance the volume variations due to elastic deformations. Let $\eta \in H^3(\Omega_S(0))^3$ be a lifting of the unit outward normal on $\partial\Omega_S(0)$. We can always suppose that $\eta \in \mathcal{E}$. Then, we define X_S by:

$$X_S(t, 0, y) = a(t) + Q(t)(y - g_0) + Q(t)(\xi_1(t, y) + \lambda(t)\eta(y)), \quad \forall y \in \Omega_S(0), \quad (4.2)$$

where $\lambda(t)$ is such that, for each $t \in [0, T]$,

$$\int_{\Omega_S(0)} \det \nabla X_S(t, 0, y) dy = \int_{\Omega_S(0)} \det (Id + \nabla \xi_1(t, y) + \lambda(t) \nabla \eta(y)) dy = \operatorname{vol}(\Omega_S(0)).$$

If the flow $X_S(t, 0, \cdot)$ is invertible from $\Omega_S(0)$ to $\Omega_S(t) := X_S(t, 0, \Omega_S(0))$, for each $t \in [0, T]$, this condition is equivalent to the conservation of the global volume of the solid domain. Thanks to the following lemma, the parameter λ is well defined for ξ_1 small enough:

Lemma 4.1. *There exists $\rho_1 > 0$ and $\rho_2 > 0$ and a mapping*

$$\phi : B(0, \rho_2) \subset H^3(\Omega_S(0))^3 \mapsto B(0, \rho_1) \subset \mathbb{R},$$

such that, for each $\xi_1 \in B(0, \rho_2)$, there exists a unique $\lambda = \phi(\xi_1) \in B(0, \rho_1)$ satisfying

$$\int_{\Omega_S(0)} \det(\text{Id} + \nabla \xi_1(y) + \lambda \nabla \eta(y)) \, dy = \text{vol}(\Omega_S(0)). \quad (4.3)$$

Proof. To prove this result, we apply the implicit function theorem on the function

$$\begin{aligned} f : H^3(\Omega_S(0))^3 \times \mathbb{R} &\mapsto \mathbb{R} \\ f : (\xi_1, \lambda) &\mapsto \int_{\Omega_S(0)} \det(\text{Id} + \nabla \xi_1(y) + \lambda \nabla \eta(y)) \, dy. \end{aligned}$$

The function f is of class \mathcal{C}^1 on a neighbourhood of $(0, 0)$. We notice that:

$$\begin{aligned} f(0, 0) &= \text{vol}(\Omega_S(0)) \text{ and} \\ \partial_\lambda f(0, 0) &= \int_{\Omega_S(0)} \text{div} \, \eta(y) \, dy = \int_{\partial\Omega_S(0)} \eta(y) \cdot N_y \, d\gamma_y = a(\partial\Omega_S(0)) > 0, \end{aligned}$$

where $a(\partial\Omega_S(0))$ denotes the area of $\partial\Omega_S(0)$. This allows us to apply the implicit function theorem on a neighbourhood of $(0, 0)$. \square

For ξ_1 such that $\|\xi_1\|_{L^\infty(0, T; H^3(\Omega_S(0)))} \leq \rho_2$, we define

$$\xi(t, y) = \xi_1(t, y) + \lambda(t)\eta(y), \quad \forall t \in [0, T], \quad \forall y \in \Omega_S(0),$$

where $\lambda(t) = \phi(\xi_1(t))$ is given by Lemma 4.1. The flow X_S is now compatible with an incompressible motion in the fluid domain.

Lemma 4.2. *There exist strictly positive real numbers κ_0, κ_1 and $\kappa_2 \leq \rho_2$ depending only on E_0, ϵ and the domains Ω and $\Omega_S(0)$ such that for each $(a, Q, \xi) \in Y_\kappa^1$,*

- *the flow X_S defined by (4.2), where $\lambda(t) = \phi(\xi_1(t))$ is given by Lemma 4.1, can be extended by a function $Y_{S,p}$ invertible from Ω onto Ω for each fixed $t \in [0, T]$, moreover,*

$$Y_{S,p}(t, 0, y) = y, \text{ for each } t \in [0, T], \text{ for each } y \in \partial\Omega;$$

- *for each $t \in [0, T]$, $\xi(t)$ belongs to \mathcal{Y}_δ ;*
- *the distance between $\Omega_S(t) = X_S(t, 0, \Omega_S(0))$ and the boundary of Ω satisfies*

$$d(\Omega_S(t), \partial\Omega) \geq d_0. \quad (4.4)$$

Proof. Let us introduce a linear extension operator \mathcal{P} such that

$$\begin{aligned} \mathcal{P} : H^3(\Omega_S(0))^3 &\mapsto H^3(\Omega)^3 \cap H_0^1(\Omega)^3 \\ W^{1,\infty}(\Omega_S(0))^3 &\mapsto W^{1,\infty}(\Omega)^3 \cap H_0^1(\Omega)^3. \end{aligned} \quad (4.5)$$

Then, thanks to this operator, we can define on Ω an extension of $X_S(t, 0, \cdot)$, for each $t \in [0, T]$ by

$$Y_{S,p}(t, 0, \cdot) = Id + \mathcal{P}(X_S(t, 0, \cdot) - Id).$$

Thus, we have

$$\|Y_{S,p}(t, 0, \cdot) - Id\|_{L^\infty(0,T;W^{1,\infty}(\Omega))^3} \leq C_{\mathcal{P}} \|X_S(t, 0, \cdot) - Id\|_{L^\infty(0,T;W^{1,\infty}(\Omega_S(0)))^3}, \quad (4.6)$$

where $C_{\mathcal{P}} \geq 1$ is the continuity constant of \mathcal{P} .

Let us suppose that the neighbourhood introduced in Lemma 4.1 are small enough to have

$$C_0 \rho_2 + \|\eta\|_{W^{1,\infty}(\Omega_S(0))^3} \rho_1 \leq \frac{e_0}{3C_{\mathcal{P}}}, \quad (4.7)$$

where C_0 is the continuity constant of the imbedding $H^3(\Omega_S(0)) \hookrightarrow W^{1,\infty}(\Omega_S(0))$ and e_0 the constant given in Lemma 2.4 associated to the domain Ω . In this case, we have

$$\|X_S(t, 0, \cdot) - a(t) - Q(t)(y - g_0)\|_{L^\infty(0,T;W^{1,\infty}(\Omega_S(0)))^3} \leq \frac{e_0}{3C_{\mathcal{P}}}.$$

Consequently,

$$\begin{aligned} &\|X_S(t, 0, y) - y\|_{L^\infty(0,T;W^{1,\infty}(\Omega_S(0)))^3} \\ &\leq \|X_S(t, 0, \cdot) - a(t) - Q(t)(y - g_0)\|_{L^\infty(0,T;W^{1,\infty}(\Omega_S(0)))^3} \\ &\quad + \|(Q(t) - Id)(y - g_0)\|_{L^\infty(0,T;W^{1,\infty}(\Omega_S(0)))^3} + \|a(t) - g_0\|_{L^\infty(0,T)} \\ &\leq \frac{e_0}{3C_{\mathcal{P}}} + \kappa_0 + \sup_{y \in \Omega_S(0)} \|y - g_0\| \kappa_1 \leq \frac{e_0}{C_{\mathcal{P}}}, \end{aligned}$$

if we choose

$$\kappa_0 \leq \frac{e_0}{3C_{\mathcal{P}}} \text{ and } \sup_{y \in \Omega} \|y - g_0\| \kappa_1 \leq \frac{e_0}{3C_{\mathcal{P}}}. \quad (4.8)$$

So, thanks to Lemma 2.4 and (4.6), we deduce that $Y_{S,p}(t, 0, \cdot)$ is injective on Ω for all $t \in [0, T]$. Moreover, since

$$\forall t \in [0, T], \forall y \in \partial\Omega, Y_{S,p}(t, 0, y) = y,$$

thanks to a connexity argument, we easily show that in fact, $Y_{S,p}(t, 0, \cdot)$ is bijective from Ω onto Ω .

A simple calculation shows that the two last properties (ii) and (iii) are satisfied for κ_0, κ_1 and $\kappa_2 \leq \rho_2$ smaller than a constant depending on ϵ, E_0, ρ_S^0 and $\Omega_S(0)$. \square

Up to now, we consider κ_0, κ_1 and κ_2 such that this lemma is valid. This lemma also obviously implies that X_S is bijective from $\Omega_S(0)$ onto $\Omega_S(t)$. This

allows us to define the eulerian velocity u_S associated to X_S . Then, we extend u_S by $u_{S,p}$ defined on Ω by the following Stokes problem:

$$\begin{cases} -\Delta u_{S,p} + \nabla q = 0 & \text{on } \Omega_F(t), \\ \operatorname{div} u_{S,p} = 0 & \text{on } \Omega_F(t), \\ u_{S,p} = u_S & \text{on } \partial\Omega_S(t), \\ u_{S,p} = 0 & \text{on } \partial\Omega \cap \partial\Omega_F(t), \end{cases}$$

with $\Omega_F(t) = \Omega \setminus \overline{\Omega_S(t)}$. This is possible thanks to (4.3) and the fact that, by (4.4), $\Omega_F(t)$ has the same regularity as $\Omega_S(t)$ for each $t \in [0, T]$. On the other hand, we can extend w_F on Ω by 0 and we define u by:

$$u(t, x) = u_{S,p}(t, x) + [(Y_{S,p}(0, t, \cdot))^* w_F](t, x), \forall (t, x) \in [0, T] \times \Omega,$$

where

$$\begin{aligned} & [(Y_{S,p}(0, t, \cdot))^* w_F](t, x) \\ &= \det \nabla Y_{S,p}(0, t, x) \nabla Y_{S,p}(t, 0, Y_{S,p}(0, t, x)) w_F(t, Y_{S,p}(0, t, x)) \end{aligned}$$

is the Piola–Kirchhoff transform. We notice that, for each $t \in [0, T]$, for each $x \in \Omega_S(t)$,

$$\operatorname{div} u(t, x) = \operatorname{div} [(Y_{S,p}(0, t, \cdot))^* w_F](t, x) = 0.$$

Finally, the existence of ρ_F , a solution of (3.4), is due to a result of di Perna and Lions ([12]):

Proposition 4.3. *Let $\rho_0 \in L^\infty((0, T) \times \Omega)$ and $v \in L^2(0, T; V(\Omega))$ be given where $V(\Omega)$ is defined by (4.1). The following problem has a unique weak solution $\rho \in L^\infty((0, T) \times \Omega) \cap \mathcal{C}(0, T; L^1(\Omega))$:*

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho v) = 0 & \text{on } \Omega, \\ \rho(0) = \rho_0 & \text{on } \Omega. \end{cases}$$

This means that there exists a unique function $\rho \in L^\infty((0, T) \times \Omega) \cap \mathcal{C}(0, T; L^1(\Omega))$ such that, $\forall \eta \in \mathcal{C}^1((0, T) \times \Omega)$,

$$\int_0^T \int_\Omega \rho (\partial_t \eta + (v \cdot \nabla) \eta) dx dt = - \int_\Omega \rho_0 \eta(0) dx.$$

This ends the representation of velocities. We can now easily check that we have constructed (u, ρ_F, a, Q, ξ) which satisfies conditions (3.1) to (3.4). By this way, we have represented a velocity compatible with solid and fluid motions. We denote Θ the mapping which maps (w_F, a, Q, ξ_1) on (u, ρ_F, a, Q, ξ) .

5. Finite-dimensional problem

We will use the previous representation of velocities to construct approximate solutions in finite dimension. First of all, we will solve a linearized finite-dimensional

problem and then we will obtain a solution of the non linear finite-dimensional problem thanks to a fixed point theorem.

5.1. Finite-dimensional linearized problem

Let $(\varphi_i)_{i \geq 1}$ be a basis of $V(\Omega_F(0))$ orthonormal in $L^2(\Omega_F(0))^3$ and $(\psi_i)_{i \geq 0}$ be an orthogonal basis of \mathcal{E} with $\psi_0 = \eta$, the lifting of the unit outward normal defined in the previous paragraph. We suppose that $(\psi_i)_{i \geq 1}$ is an orthonormal system in $H^3(\Omega_S(0))^3$.

Suppose that we have $(\tilde{w}_F^N, (\tilde{a}^N, \tilde{Q}^N, \tilde{\xi}_1^N)) \in Y^0 \times Y_\kappa^1$ defined by

$$\tilde{w}_F^N(t, \cdot) = \sum_{i=1}^N \tilde{\gamma}_i(t) \varphi_i, \quad \tilde{\xi}_1^N(t, \cdot) = \sum_{i=1}^N \tilde{\alpha}_i(t) \psi_i, \quad (5.1)$$

with $\tilde{a}^N \in \mathcal{C}^1(0, T)^3$, $\tilde{Q}^N \in \mathcal{C}^1(0, T; SO_3(\mathbb{R}))$, $(\tilde{\alpha}_i)_{1 \leq i \leq N} \in \mathcal{C}^1(0, T)^N$ and $(\tilde{\gamma}_i)_{1 \leq i \leq N} \in \mathcal{C}^0(0, T)^N$.

Then, we can construct

$$(\tilde{u}^N, \tilde{\rho}_F^N, \tilde{a}^N, \tilde{Q}^N, \tilde{\xi}^N) = \Theta(\tilde{w}_F^N, \tilde{a}^N, \tilde{Q}^N, \tilde{\xi}_1^N)$$

as explained above. To fix the notations, we recall some steps of this construction. On $\Omega_S(0)$,

$$\begin{aligned} \tilde{X}_S^N(t, 0, y) &= \tilde{a}^N(t) + \tilde{Q}^N(t)(y - g_0) + \tilde{Q}^N(t) \left(\tilde{\xi}_1^N(t, y) + \tilde{\alpha}_0(t) \eta(y) \right) \\ &= \tilde{a}^N(t) + \tilde{Q}^N(t)(y - g_0) + \tilde{Q}^N(t) \tilde{\xi}^N(t, y), \end{aligned}$$

where $\tilde{\alpha}_0$ balances the volume variations. The lagrangian flow $\tilde{X}_S^N(t, 0, \cdot)$ is extended by an invertible function $\tilde{Y}_{S,p}^N$ from Ω onto Ω . More precisely, $\tilde{X}_S^N(t, 0, \cdot)$ satisfies

$$\|\tilde{X}_S^N(t, 0, \cdot) - Id\|_{L^\infty(0, T; W^{1, \infty}(\Omega_S(0)))} \leq \frac{e_0}{C_1}. \quad (5.2)$$

The inertia tensor \tilde{J}^N is defined by (1.14), where we replace Q and ξ respectively by \tilde{Q}^N and $\tilde{\xi}^N$. At last, we define:

$$\tilde{U}_E^N(t, y) = \sum_{i=0}^N \tilde{\alpha}'_i(t) \psi_i(y) = \partial_t \tilde{\xi}^N(t, y).$$

By this way, we have

$$\begin{aligned} \tilde{u}^N(t, x) &= \frac{d\tilde{a}^N}{dt}(t) (\chi_{\tilde{\Omega}_S^N(t)})_p + (\tilde{\omega}^N(t) \wedge (x - \tilde{a}^N(t)) \chi_{\tilde{\Omega}_S^N(t)})_p \\ &\quad + \left(\tilde{Q}^N(t) \left(\sum_{i=0}^N \tilde{\alpha}'_i(t) \psi_i(\tilde{X}_S^N(0, t, x)) \right) \right)_p + \sum_{i=1}^N \tilde{\beta}_i(t) [(\tilde{Y}_{S,p}^N(0, t, \cdot))^* \varphi_i](t, x). \end{aligned} \quad (5.3)$$

We look now for a solution $(u^N, \rho_F^N, a^N, Q^N, \xi^N)$ satisfying the variational formulation given below such that:

- On $\Omega_S(0)$, $u^N(t, \tilde{X}_S^N(t, 0, y)) = \dot{a}^N(t) + \omega^N(t) \wedge (\tilde{Q}^N(t)(y - g_0 + \tilde{\xi}^N(t, y))) + \tilde{Q}^N(t)U_E^N(t, y)$ with $U_E^N(t, y) = \sum_{i=0}^N \beta_i(t)\psi_i(y)$, where $\beta_0(t)$ is such that:

$$\int_{\partial\tilde{\Omega}_S^N(t)} u^N(t, x) \cdot n_x d\gamma(x) = 0; \quad (5.4)$$

- $u^N(t, x) = \dot{a}^N(t)(\chi_{\tilde{\Omega}_S^N(t)})_p + (\omega^N(t) \wedge (x - \tilde{a}^N(t))\chi_{\tilde{\Omega}_S^N(t)})_p + (\tilde{Q}^N(t)U_E^N(t, \tilde{X}_S^N(0, t, x)))_p + \sum_{i=1}^N \gamma_i(t)[(\tilde{Y}_{S,p}^N(0, t, \cdot))^* \varphi_i](t, x)$ (5.5)

on $\tilde{\Omega}_F^N(t)$;

- $\xi^N(t, \cdot) = \sum_{i=0}^N \alpha_i(t)\psi_i$ on $\Omega_S(0)$; (5.6)

- $$\begin{cases} \partial_t \rho_F^N + \operatorname{div}(\rho_F^N u^N) = 0 & \text{on } \Omega, \\ \rho_F(0) = \rho_F^0 \chi_{\Omega_F(0)} & \text{on } \Omega. \end{cases} \quad (5.7)$$

Remark 5.1. The condition (5.4) of volume conservation is now linear. Indeed, it is equivalent to

$$\sum_{i=1}^N \tilde{r}_i^N(t)\beta_i(t) + \tilde{r}_0^N(t)\beta_0(t) = 0, \quad (5.8)$$

with $\forall 0 \leq i \leq N, \forall t \in [0, T]$,

$$\tilde{r}_i^N(t) = \int_{\partial\tilde{\Omega}_S^N(t)} [\tilde{Q}^N(t)\psi_i(\tilde{X}_S^N(0, t, x))] \cdot n_x d\gamma_x. \quad (5.9)$$

We recall that the unit outward normal n_x on $\partial\tilde{\Omega}_S^N(t)$ satisfies:

$$n_{\tilde{X}_S^N(t, 0, y)} = \frac{\operatorname{cof} \nabla \tilde{X}_S^N(t, 0, y) N_y}{\|\operatorname{cof} \nabla \tilde{X}_S^N(t, 0, y) N_y\|}, \quad \forall y \in \partial\Omega_S(0). \quad (5.10)$$

Therefore, thanks to (5.2), we check that: $\tilde{r}_0^N(t) \geq r > 0, \forall t \in [0, T]$. Thus, for each $t \in [0, T]$, $\beta_0(t)$ is well defined and we can write that

$$U_E^N(t, y) = \sum_{i=1}^N \beta_i(t)\tilde{\psi}_i^1(t, y) \text{ with } \tilde{\psi}_i^1(t, y) = \psi_i(y) - \frac{\tilde{r}_i^N(t)}{\tilde{r}_0^N(t)}\eta(y).$$

As $\tilde{\psi}_i^1$ verifies

$$\int_{\partial\tilde{\Omega}_S^N(t)} [\tilde{Q}^N(t)\tilde{\psi}_i^1(t, \tilde{X}_S^N(0, t, x))] \cdot n_x d\gamma_x = 0,$$

we can extend this function by $\tilde{\Psi}_i$ defined on $\tilde{\Omega}_S^N(t)$ by a Stokes problem

$$\tilde{\Psi}_i = (\tilde{Q}^N(t)\tilde{\psi}_i^1(t, \tilde{X}_S^N(0, t, \cdot)))_p.$$

So, u^N can be rewritten as:

$$\begin{aligned} u^N(t, x) &= \dot{a}^N(t)(\chi_{\tilde{\Omega}_S^N(t)})_p + (\omega^N(t) \wedge (x - \tilde{a}^N(t))\chi_{\tilde{\Omega}_S^N(t)})_p \\ &\quad + \sum_{i=1}^N \beta_i(t)\tilde{\Psi}_i(t, x) + \sum_{i=1}^N \gamma_i(t)[(\tilde{Y}_{S,p}^N(0, t, \cdot))^* \varphi_i](t, x). \end{aligned} \quad (5.11)$$

Moreover, we also linearize the variational formulation around the fixed trajectory. The solution $(u^N, a^N, Q^N, \xi^N, U_E^N)$ has to satisfy:

$$\begin{aligned} & m \int_0^T \ddot{u}^N \cdot \dot{b}^N dt + \int_0^T \frac{d}{dt} (\tilde{J}^N \omega^N) \cdot r^N dt \\ & + \int_0^T \int_{\Omega_S(0)} \rho_S^0 (\tilde{\omega}^N \wedge \tilde{Q}^N (y - g_0 + \tilde{\xi}^N)) \wedge (\omega^N \wedge \tilde{Q}^N (y - g_0 + \tilde{\xi}^N)) \cdot r^N dy dt \\ & - \int_0^T \int_{\Omega_S(0)} \rho_S^0 (\tilde{Q}^N \partial_t \tilde{\xi}^N \wedge (\omega^N \wedge \tilde{Q}^N (y - g_0 + \tilde{\xi}^N))) \cdot r^N dy dt \\ & + \int_0^T \int_{\Omega_S(0)} \rho_S^0 \tilde{Q}^N (y - g_0 + \tilde{\xi}^N) \wedge (\tilde{\omega}^N \wedge \tilde{Q}^N \partial_t \xi^N) \cdot r^N dy dt \\ & + \int_0^T \int_{\Omega_S(0)} \rho_S^0 \tilde{Q}^N (\tilde{\xi}^N \wedge \partial_t U_E^N) \cdot r^N dy dt + \int_0^T \int_{\Omega_S(0)} \rho_S^0 \partial_t U_E^N \cdot V_E^N dy dt \\ & + \int_0^T \int_{\Omega_S(0)} \rho_S^0 ((\tilde{Q}^N)^{-1} \tilde{\omega}^N \wedge \partial_t \xi^N) \cdot V_E^N dy dt \\ & + \int_0^T \int_{\Omega_S(0)} \rho_S^0 ((\tilde{Q}^N)^{-1} \omega^N \wedge \partial_t \tilde{\xi}^N) \cdot V_E^N dy dt \\ & + \int_0^T \int_{\Omega_S(0)} \rho_S^0 ((\tilde{Q}^N)^{-1} \dot{\omega}^N \wedge \tilde{\xi}^N) \cdot V_E^N dy dt \\ & + \int_0^T \int_{\Omega_S(0)} \rho_S^0 ((\tilde{Q}^N)^{-1} \omega^N \wedge ((\tilde{Q}^N)^{-1} \tilde{\omega}^N \wedge (y - g_0 + \tilde{\xi}^N))) \cdot V_E^N dy dt \\ & + \epsilon \int_0^T (\partial_t U_E^N, V_E^N)_{H^3(\Omega_S(0))} dt + \int_0^T \int_{\Omega_S(0)} \Sigma_E^2(\xi^N) : \varepsilon_y(V_E^N) dy dt \\ & + \int_0^T \int_{\Omega} \tilde{\rho}_F^N \partial_t u^N \cdot v^N dx dt + \int_0^T \int_{\Omega} \tilde{\rho}_F^N ((\tilde{u}^N \cdot \nabla) u^N) \cdot v^N dx dt \end{aligned}$$

$$\begin{aligned}
& + 2\nu \int_0^T \int_{\tilde{\Omega}_F^N(t)} \varepsilon_x(u^N) : \varepsilon_x(v^N) dx dt + \int_0^T (\partial_t \xi^N, \eta^N)_{H^3(\Omega_S(0))} dt \\
& - \int_0^T (U_E^N, \eta^N)_{H^3(\Omega_S(0))} dt = 0
\end{aligned}$$

for all

$$\begin{aligned}
v^N(t, x) &= \dot{b}^N(t) (\chi_{\tilde{\Omega}_S^N(t)})_p + (r^N(t) \wedge (x - \tilde{a}^N(t)) \chi_{\tilde{\Omega}_S^N(t)})_p \\
& + \sum_{i=1}^N B_i(t) \tilde{\Psi}_i(t, x) + \sum_{i=1}^N C_i(t) [(\tilde{Y}_{S,p}^N(0, t, \cdot))^* \varphi_i](t, x)
\end{aligned}$$

and

$$V_E^N(t, y) = \sum_{i=1}^N B_i(t) \tilde{\psi}_i^1(t, y), \quad \eta^N(t, y) = \sum_{i=0}^N A_i(t) \psi_i(y)$$

with $b^N \in H^1(0, T)^3$, $r^N \in L^2(0, T)^3$, A_i , B_i and $C_i \in L^2(0, T)$.

We now have a classical problem in the unknowns a^N , Q^N , and α_i , β_i and γ_i . We can solve this linear ordinary differential system. Indeed, the ordinary differential system we solve is of the type

$$A^N(t) \dot{Y}^N(t) = M^N(t) Y^N(t) \text{ with } A^N, M^N \in \mathcal{C}(0, T; \mathcal{M}_{3N+7}(\mathbb{R})) \quad (5.12)$$

with $Y^N = (\dot{a}^N, \omega^N, \alpha_0, \dots, \alpha_N, \beta_1, \dots, \beta_N, \gamma_1, \dots, \gamma_N)^t$.

As $\tilde{\xi}^N(t)$ belongs to \mathcal{Y}_δ , we can prove that the matrix A^N is a symmetric positive definite matrix. We can also notice that $\beta_i = \alpha'_i$, for each $1 \leq i \leq N$. Thus, we have equivalently

$$U_E^N(t, y) = \partial_t \xi^N(t, y), \quad \forall t \in [0, T], \quad \forall y \in \Omega_S(0).$$

At last, the existence and uniqueness of ρ^N , a solution of (5.7) results from proposition 4.3. We obtained a unique solution $(u^N, \rho_F^N, a^N, Q^N, \xi^N)$ that satisfies our linear problem in finite dimension.

5.2. Fixed point argument

Moreover, we can see that our solution satisfies the energy estimate

$$\begin{aligned}
& \frac{1}{2} \int_{\Omega_S(0)} \rho_S^0(y) \left| \dot{a}^N(t) + \omega^N(t) \wedge \tilde{Q}^N(t)(y - g_0 + \tilde{\xi}^N(t, y)) + \tilde{Q}^N(t) \partial_t \xi^N(t, y) \right|^2 dy \\
& + \frac{1}{2} \int_{\Omega} \tilde{\rho}_F^N(t, x) |u_F^N(t, x)|^2 dx + 2\nu \int_0^t \int_{\tilde{\Omega}_F^N(s)} |\varepsilon_x(u_F^N(s, x))|^2 dx ds \\
& + \frac{1}{2} \epsilon \left\| \partial_t \xi^N(t, \cdot) \right\|_{H^3(\Omega_S(0))}^2 + \frac{1}{2} \int_{\Omega_S(0)} \Sigma_E^2(\xi^N)(t, y) : \varepsilon_y(\xi^N)(t, y) dy \leq E_0^N \leq 2E_0,
\end{aligned} \quad (5.13)$$

for N large enough, where E_0^N is the initial energy in finite dimension. As $\tilde{\xi}^N(t)$ belongs to \mathcal{Y}_δ for each $t \in [0, T]$, we deduce from this inequality that:

$$|\dot{a}^N(t)|^2 + |\omega^N(t)|^2 + \sum_{i=0}^N \alpha_i(t)^2 + \sum_{i=0}^N \dot{\alpha}_i(t)^2 + \sum_{i=1}^N \gamma_i(t)^2 \leq M, \quad (5.14)$$

where M is a constant depending on ϵ , E_0 , $\Omega_S(0)$, ρ_S^0 and δ . We define the set \mathcal{A} by

$$\mathcal{A} = \mathcal{C}^1(0, T)^3 \times \mathcal{C}^1(0, T; SO_3(\mathbb{R})) \times \mathcal{C}^1(0, T)^N \times \mathcal{C}(0, T)^N,$$

with the usual norm and the set \mathcal{C} by:

$$\mathcal{C} = \left\{ Z = (a^N, Q^N, \alpha_1, \dots, \alpha_N, \gamma_1, \dots, \gamma_N) \in \mathcal{A} / \|Z\|_{\mathcal{A}} \leq M \text{ and} \right. \\ \left. \|a^N - g_0\|_{L^\infty(0, T)} \leq \kappa_0, \|Q^N - Id\|_{L^\infty(0, T)} \leq \kappa_1, \sup_{0 \leq t \leq T} \sum_{i=1}^N \alpha_i(t)^2 \leq \kappa_2^2 \right\}.$$

The set \mathcal{C} is a non empty closed convex set of \mathcal{A} . We also define the operator K by

$$\begin{aligned} K : \mathcal{C} &\mapsto \mathcal{A} \\ \tilde{Z}^N &\mapsto Z^N \\ (\tilde{a}^N, \tilde{Q}^N, \tilde{\alpha}_1, \dots, \tilde{\alpha}_N, \tilde{\gamma}_1, \dots, \tilde{\gamma}_N) &\mapsto (a^N, Q^N, \alpha_1, \dots, \alpha_N, \gamma_1, \dots, \gamma_N), \end{aligned}$$

where \tilde{Z}^N and Z^N are the coefficients appearing in the writing of \tilde{u}^N (see (5.3)) and u^N (see (5.11)). We notice that K is continuous.

Then, in order to apply Schauder's fixed point theorem on the operator K , we need to keep small rigid displacements and small elastic deformations i.e.

$$\|a^N - g_0\|_{L^\infty(0, T)} \leq \kappa_0, \|Q^N - Id\|_{L^\infty(0, T)} \leq \kappa_1, \sup_{0 \leq t \leq T} \sum_{i=1}^N \alpha_i(t)^2 \leq \kappa_2^2.$$

Thanks to (5.14), these estimates are verified until $T = T_0$ with T_0 depending on E_0 , ϵ , κ_0 , κ_1 , κ_2 and δ . Thus, on this interval $[0, T_0]$, $K(\mathcal{C}) \subset \mathcal{C}$. From now on, we will prove the existence of a solution on $[0, T_0]$. The last section will be devoted to the extension of the solution to the interval $[0, T]$ where T is independent of κ_0 , κ_1 and κ_2 as defined in Theorem 3.4.

Now, for the compactness of K , using Proposition 2.2 and Proposition 2.3, we prove that, in (5.12), $(A^N)^{-1}$ and M^N are bounded in $L^\infty(0, T_0)$ uniformly in $\tilde{Z}^N \in \mathcal{C}$. Therefore, we obtain that

$$\left\| \frac{dZ^N}{dt} \right\|_{L^\infty(0, T_0)} \leq C, \forall Z^N \in K(\mathcal{C}).$$

And so, thanks to Ascoli's theorem, K is a compact operator and as \mathcal{C} is bounded, $\overline{K(\mathcal{C})}$ is compact.

Thus, we can apply Schauder's theorem and we obtain the existence of a fixed point of K . We easily verify that this fixed point satisfies

$$(u^N, \rho_F^N, a^N, Q^N, \xi^N) = \Theta(w_F^N, a^N, Q^N, \xi_1^N),$$

with:

$$w_F^N(t, \cdot) = \sum_{i=1}^N \gamma_i(t) \varphi_i, \quad \xi_1^N(t, \cdot) = \sum_{i=1}^N \alpha_i(t) \psi_i.$$

Moreover our fixed point is a solution of the finite-dimensional approximation of the non linear problem

$$\begin{aligned} & \int_0^{T_0} \int_{\Omega_S(0)} \rho_S^0(y) \partial_t^2 X_S^N(t, 0, y) V^N(t, y) dy dt \\ & + \epsilon \int_0^{T_0} (\partial_t^2 \xi^N, V_E^N)_{H^3(\Omega_S(0))} dt \\ & + \int_0^{T_0} \int_{\Omega_S(0)} \Sigma_E^2(\xi^N) : \varepsilon_y(V_E^N) dy dt + \int_0^{T_0} \int_{\Omega} \rho_F^N \partial_t u^N \cdot v^N dx dt \quad (5.15) \\ & + \int_0^{T_0} \int_{\Omega} \rho_F^N ((u^N \cdot \nabla) u^N) \cdot v^N dx dt \\ & + 2\nu \int_0^{T_0} \int_{\Omega_F^N(t)} \varepsilon_x(u^N) : \varepsilon_x(v^N) dx dt = 0, \end{aligned}$$

for all V^N, v^N such that, for any $(t, y) \in [0, T_0] \times \Omega_S(0)$,

$$V^N(t, y) = \dot{b}^N(t) + r^N(t) \wedge (Q^N(t)(y - g_0 + \xi^N(t, y))) + Q^N(t) V_E^N(t, y),$$

where

$$V_E^N(t, y) = \sum_{i=1}^N B_i(t) \psi_i^1(t, y),$$

and, for any $(t, x) \in [0, T_0] \times \Omega$,

$$v^N(t, x) = (V^N(t, X_S^N(0, t, x)))_p + \sum_{i=1}^N C_i(t) [(Y_{S,p}^N(0, t, \cdot))^* \varphi_i](t, x),$$

with $b^N \in H^2(0, T)^3$, $r^N \in H^1(0, T_0)^3$ and $B_i, C_i \in H^1(0, T_0)^3$ equal to 0 for $t = T_0$.

The energy estimate satisfied by our solution is: $\forall t \in [0, T_0]$,

$$\begin{aligned}
& \frac{1}{2} \int_{\Omega_S(0)} \rho_S^0(y) \left| \dot{a}^N(t) + \omega^N(t) \wedge Q^N(t)(y - g_0 + \xi^N(t)) + Q^N(t) \partial_t \xi^N(t) \right|^2 dy \\
& + \frac{1}{2} \int_{\Omega} \rho_F^N(t, x) \left| u_F^N(t, x) \right|^2 dx + 2\nu \int_0^t \int_{\Omega_F^N(s)} |\varepsilon_x(u_F^N(s, x))|^2 dx ds \\
& + \frac{1}{2} \varepsilon \left\| \partial_t \xi^N(t, \cdot) \right\|_{H^3(\Omega_S(0))}^2 \\
& + \frac{1}{2} \int_{\Omega_S(0)} \Sigma_E^2(\xi^N)(t, y) : \varepsilon_y(\xi^N)(t, y) dy \leq E_0^N \leq 2E_0.
\end{aligned} \tag{5.16}$$

From this inequality and the fact that $\xi^N(t)$ belongs to \mathcal{Y}_δ , for each $t \in [0, T]$, we deduce the following weak convergences:

Proposition 5.2. *We can extract from (a^N) , (ω^N) , (u^N) , (ξ^N) and (X^N) subsequences (still written (a^N) , (ω^N) , (u^N) , (ξ^N) and (X^N)) such that*

$$\begin{aligned}
a^N & \rightharpoonup a & \text{in } W^{1,\infty}(0, T_0)^3 w^*, \\
\omega^N & \rightharpoonup \omega & \text{in } L^\infty(0, T_0)^3 w^*, \\
u^N & \rightharpoonup u & \text{in } L^\infty(0, T_0; L^2(\Omega))^3 w^* \text{ and} \\
& & \text{in } L^2(0, T_0; H_0^1(\Omega))^3 w, \\
\xi^N & \rightharpoonup \xi & \text{in } L^2(0, T_0; H^1(\Omega_S(0)))^3 w, \\
\partial_t \xi^N & \rightharpoonup \partial_t \xi & \text{in } L^\infty(0, T_0; H^3(\Omega_S(0)))^3 w^*, \\
X_S^N & \rightharpoonup X_S & \text{in } W^{1,\infty}(0, T_0; H^3(\Omega_S(0)))^3 w^*,
\end{aligned}$$

where X_S is defined on $\Omega_S(0)$ by

$$X_S(t, 0, y) = a(t) + Q(t)(y - g_0) + Q(t)\xi(t, y), \quad \forall t \in [0, T_0], \quad \forall y \in \Omega_S(0).$$

6. Compactness results

6.1. Idea of the proof

In order to pass to the limit when N goes to infinity, we have to show further compactness results.

The strong convergence of $(\rho_F^N)_{N \in \mathbb{N}}$ is obtained directly by using a result of di Perna and Lions (see [19]). This result involves the strong convergence of (ρ_F^N) in $\mathcal{C}(0, T_0; L^p(\Omega))$ for all $1 \leq p < +\infty$ to $\rho_F \in L^\infty((0, T_0) \times \Omega) \cap \mathcal{C}(0, T_0; L^1(\Omega))$, the unique solution (see Proposition 4.3) of

$$\begin{cases} \partial_t \rho_F + \operatorname{div}(\rho_F u) = 0 & \text{on } \Omega, \\ \rho_F(0) = \rho_F^0 \chi_{\Omega_F(0)} & \text{on } \Omega. \end{cases}$$

Moreover, an auxiliary result proven in [19] is

$$\sqrt{\rho_F^N} \rightarrow \sqrt{\rho_F} \text{ in } \mathcal{C}(0, T_0; L^p(\Omega)), \quad \forall 1 \leq p < +\infty. \tag{6.1}$$

In order to obtain a compactness result on the velocities, we will use the following result (see [25]) which characterizes compact sets:

Lemma 6.1. *Let B be a Banach space and $F \hookrightarrow L^p(0, T_0; B)$ with $1 \leq p < \infty$. Then F is a relatively compact set of $L^p(0, T_0; B)$ if and only if*

- $\left\{ \int_{t_1}^{t_2} f(t) dt, f \in F \right\}$ is relatively compact in B for each $0 < t_1 < t_2 < T_0$,
- $\|f(t+h) - f(t)\|_{L^p(0, T_0-h; B)} \rightarrow 0$ when $h \rightarrow 0$, uniformly for $f \in F$.

We will show that $\left(\sqrt{\rho_F^N} u^N\right)$ converges strongly in $L^2((0, T_0) \times \Omega)$ to $\sqrt{\rho_F} u$.

It is clear, using (6.1), that $\left(\sqrt{\rho_F^N} u^N\right)$ converges weakly in $L^2((0, T_0) \times \Omega)$ to $\sqrt{\rho_F} u$. First, let us prove the following lemma which is the first step of the proof:

Lemma 6.2. $\forall 0 < t_1 < t_2 < T_0$,

$$\lim_{N \rightarrow \infty} \int_{t_1}^{t_2} \sqrt{\rho_F^N(t)} u^N(t) dt = \int_{t_1}^{t_2} \sqrt{\rho_F(t)} u(t) dt \text{ in } L^2(\Omega).$$

Proof. We see that

$$\begin{aligned} & \int_{t_1}^{t_2} \left(\sqrt{\rho_F^N(t)} u^N(t) - \sqrt{\rho_F(t)} u(t) \right) dt \\ &= \int_{t_1}^{t_2} \sqrt{\rho_F(t)} (u^N(t) - u(t)) dt + \int_{t_1}^{t_2} \left(\sqrt{\rho_F^N(t)} - \sqrt{\rho_F(t)} \right) u^N(t) dt. \end{aligned} \quad (6.2)$$

Choosing a subdivision a_i , $1 \leq i \leq P$, of $[t_1, t_2]$, we can write the first term of the sum in (6.2) as

$$\begin{aligned} & \int_{t_1}^{t_2} \sqrt{\rho_F(t)} (u^N(t) - u(t)) dt \\ &= \sum_{i=1}^P \int_{a_i}^{a_{i+1}} \sqrt{\rho_F(a_{i+\frac{1}{2}})} (u^N(t) - u(t)) dt \\ & \quad + \sum_{i=1}^P \int_{a_i}^{a_{i+1}} \left(\sqrt{\rho_F(t)} - \sqrt{\rho_F(a_{i+\frac{1}{2}})} \right) (u^N(t) - u(t)) dt, \end{aligned} \quad (6.3)$$

where $a_{i+\frac{1}{2}}$ is the middle of the interval $[a_i, a_{i+1}]$.

Since (u^N) is bounded in $L^2(0, T_0; H_0^1(\Omega))^3 \cap L^\infty(0, T_0; L^2(\Omega))^3$, (u^N) is bounded in $L^4((0, T_0); L^3(\Omega))^3$ (see [18]) and thus in $L^3((0, T_0) \times \Omega)^3$. Therefore, thanks to Cauchy–Schwarz inequality and Hölder inequality, we have

$$\left\| \sum_{i=1}^P \int_{a_i}^{a_{i+1}} \left(\sqrt{\rho_F(t)} - \sqrt{\rho_F(a_{i+\frac{1}{2}})} \right) (u^N(t) - u(t)) dt \right\|_{L^2(\Omega)}^2$$

$$\begin{aligned}
&\leq C \int_{\Omega} \left(\sum_{i=1}^P \int_{a_i}^{a_{i+1}} \left| \sqrt{\rho_F(t)} - \sqrt{\rho_F(a_{i+\frac{1}{2}})} \right|^2 |u^N(t) - u(t)|^2 dt \right) dx \\
&\leq C \|u^N - u\|_{L^3((0,T)\times\Omega)}^2 \left(\sum_{i=1}^P \int_{a_i}^{a_{i+1}} \int_{\Omega} \left| \sqrt{\rho_F(t)} - \sqrt{\rho_F(a_{i+\frac{1}{2}})} \right|^6 dx dt \right)^{\frac{1}{6}} \\
&\leq C \left(\sum_{i=1}^P \int_{a_i}^{a_{i+1}} \int_{\Omega} \left| \sqrt{\rho_F(t)} - \sqrt{\rho_F(a_{i+\frac{1}{2}})} \right|^6 dx dt \right)^{\frac{1}{6}}.
\end{aligned}$$

Moreover, for $v > 0$ fixed, as $\sqrt{\rho_F}$ belongs to $\mathcal{C}(0, T_0; L^6(\Omega))$, we can find a sufficiently refined subdivision $(a_i)_{1 \leq i \leq P}$ of $[t_1, t_2]$ so that

$$\sup_{1 \leq i \leq P, a_i \leq t \leq a_{i+1}} \left\| \sqrt{\rho_F(t)} - \sqrt{\rho_F(a_{i+\frac{1}{2}})} \right\|_{L^6(\Omega)} \leq \frac{v}{C}.$$

Therefore

$$\left\| \sum_{i=1}^P \int_{a_i}^{a_{i+1}} (\sqrt{\rho_F(t)} - \sqrt{\rho_F(a_{i+\frac{1}{2}})}) (u^N(t) - u(t)) dt \right\|_{L^2(\Omega)}^2 \leq v.$$

The subdivision being fixed, we now have for the first term of the sum in (6.3)

$$\sum_{i=1}^P \int_{a_i}^{a_{i+1}} \sqrt{\rho_F(a_{i+\frac{1}{2}})} (u^N(t) - u(t)) dt = \sum_{i=1}^P \sqrt{\rho_F(a_{i+\frac{1}{2}})} \int_{a_i}^{a_{i+1}} (u^N(t) - u(t)) dt.$$

As (u^N) is bounded in $L^2(0, T_0; H_0^1(\Omega))^3$, $\left(\int_{a_i}^{a_{i+1}} u^N(t) dt \right)$ is bounded in $H_0^1(\Omega)^3$ and converges to $\int_{a_i}^{a_{i+1}} u(t) dt$ in $L^2(\Omega)^3$. So, for v fixed, for N large enough,

$$\left\| \int_{t_1}^{t_2} \sqrt{\rho_F(t)} (u^N(t) - u(t)) dt \right\|_{L^2(\Omega)} \leq v.$$

On the other hand, the second term of the sum in (6.2) satisfies:

$$\begin{aligned}
&\left\| \int_{t_1}^{t_2} (\sqrt{\rho_F^N(t)} - \sqrt{\rho_F(t)}) u^N(t) dt \right\|_{L^2(\Omega)}^2 \\
&\leq C \left\| \sqrt{\rho_F^N(t)} - \sqrt{\rho_F(t)} \right\|_{L^{\frac{3}{2}}((0,T)\times\Omega)}^2 \|u^N\|_{L^3((0,T)\times\Omega)}^2,
\end{aligned}$$

and thus, according to (6.1), $\left(\int_{t_1}^{t_2} (\sqrt{\rho_F^N(t)} - \sqrt{\rho_F(t)}) u^N(t) dt \right)$ converges to 0 in $L^2(\Omega)$. Thus, t_1 and t_2 being fixed, $\left(\int_{t_1}^{t_2} \sqrt{\rho_F^N(t)} u^N(t) dt \right)$ strongly converges to $\int_{t_1}^{t_2} \sqrt{\rho_F(t)} u(t) dt$ in $L^2(\Omega)$; this ends the proof of Lemma 6.2. \square

Now, in order to apply Lemma 6.1, we need to show that: $\forall N \in \mathbb{N}$,

$$\int_0^{T_0-h} \int_{\Omega} \left| \sqrt{\rho_F^N(t+h)} u^N(t+h) - \sqrt{\rho_F^N(t)} u^N(t) \right|^2 dx dt \leq g(h), \quad (6.4)$$

with $\lim_{h \rightarrow 0} g(h) = 0$.

This will be a consequence of the following result:

Lemma 6.3. $\exists \alpha > 0, \forall N \in \mathbb{N}$,

$$\int_0^{T_0-h} \int_{\Omega} (\rho_F^N(t+h) u^N(t+h) - \rho_F^N(t) u^N(t)) (u^N(t+h) - u^N(t)) dx dt \leq h^\alpha.$$

Indeed, using the fact that (u^N) is bounded in $L^3((0, T_0) \times \Omega)^3$ and that $\sqrt{\rho_F^N}$ converges to $\sqrt{\rho_F}$ in $\mathcal{C}(0, T_0; L^3(\Omega))$, we easily prove that Lemma 6.3 implies (6.4).

6.2. Proof of Lemma 6.3

We have to estimate:

$$A^N = \int_0^{T_0-h} \int_{\Omega} \left(\int_t^{t+h} \partial_s (\rho_F^N u^N)(s, x) ds \right) (u^N(t+h, x) - u^N(t, x)) dx dt.$$

First, we introduce the test functions

$$w_1^N(t, s, x) = \dot{a}^N(t) (\chi_{\Omega_S^N(s)})_p + (\omega^N(t) \wedge (x - a^N(s)) \chi_{\Omega_S^N(s)})_p + \left(Q^N(s) \left[\sum_{i=1}^N \alpha_i'(t) \psi_i(X_S^N(0, s, x)) + \beta_0(t, s) \psi_0(X_S^N(0, s, x)) \right] \right)_p,$$

where β_0 is defined on $[0, T]^2$ by

$$\sum_{i=1}^N r_i^N(s) \alpha_i'(t) + r_0^N(s) \beta_0(t, s) = 0, \quad \forall (t, s) \in [0, T]^2 \quad (6.5)$$

(for the definition of r_i^N for each $0 \leq i \leq N$, we refer to (5.9) where we replace \tilde{Q}^N by Q^N and $\tilde{X}^N(0, t, \cdot)$ by $X^N(0, t, \cdot)$) and

$$w_2^N(t, s, x) = \sum_{i=1}^N \gamma_i(t) \left[(Y_{S,p}^N(0, s, \cdot))^* \varphi_i \right] (s, x).$$

We can write A^N as

$$A^N = A_1^N + A_2^N,$$

where:

$$A_i^N = \int_0^{T_0-h} \int_{\Omega} \left(\int_t^{t+h} \partial_s(\rho_F^N u^N)(s, x) ds \right) (w_i^N(t+h, t+h, x) - w_i^N(t, t, x)) dx dt.$$

For each $1 \leq i \leq 2$, we have

$$A_i^N = F_i^N + R_{1,i}^N - R_{2,i}^N,$$

with

$$\begin{aligned} F_i^N &= \int_0^{T_0-h} \int_{\Omega} \int_t^{t+h} \partial_s(\rho_F^N u^N)(s, x) (w_i^N(t+h, s, x) - w_i^N(t, s, x)) ds dx dt \\ R_{1,i}^N &= \int_0^{T_0-h} \int_{\Omega} \int_t^{t+h} \partial_s(\rho_F^N u^N)(s, x) (w_i^N(t+h, t+h, x) - w_i^N(t+h, s, x)) ds dx dt \\ R_{2,i}^N &= \int_0^{T_0-h} \int_{\Omega} \int_t^{t+h} \partial_s(\rho_F^N u^N)(s, x) (w_i^N(t, t, x) - w_i^N(t, s, x)) ds dx dt. \end{aligned}$$

The terms F_i^N will be estimated later on thanks to the variational formulation. We will first show that the remaining terms $R_{1,i}^N$ and $R_{2,i}^N$ satisfy

$$R_{1,i}^N \leq C_{1,i} h, \quad R_{2,i}^N \leq C_{2,i} h \quad \text{uniformly in } N.$$

For all these estimates, we will need the following lemma.

Lemma 6.4. $\forall N \in \mathbb{N}$,

$$\forall 1 \leq i \leq 2, \quad \int_0^{T_0-h} \int_{\Omega} \int_t^{t+h} |\partial_s w_i^N(t, s, x)|^2 ds dx dt \leq Ch, \quad (6.6)$$

$$\int_0^{T_0-h} \int_{\Omega} \int_t^{t+h} |\nabla w_1^N(t, s, x)|^3 ds dx dt \leq Ch, \quad (6.7)$$

$$\int_0^{T_0-h} \left(\int_t^{t+h} \left(\int_{\Omega} |\nabla w_2^N(t, s, x)|^2 dx \right)^2 ds \right)^{\frac{1}{2}} dt \leq C. \quad (6.8)$$

Proof. The first estimate for $i = 1$ results directly from energy inequality (5.16) and from Proposition 2.3.

For $i = 2$, as (u^N) is bounded in $L^2(0, T_0; H_0^1(\Omega))^3$,

$$\sum_{i=1}^N \gamma_i(t) \varphi_i(Y_{S,p}^N(0, t, \cdot))$$

is also bounded in $L^2(0, T_0; H_0^1(\Omega))^3 \hookrightarrow L^2(0, T_0; L^6(\Omega))^3$ and

$$\sum_{i=1}^N \gamma_i(t) \nabla \varphi_i(Y_{S,p}^N(0, t, \cdot))$$

is bounded in $L^2((0, T_0) \times \Omega)^3$. Since $(Y_{S,p}^N)$ is also bounded in $W^{1,\infty}(0, T_0; H^3(\Omega))^3$, we obtain the first estimate for $i = 2$. The second estimate is shown using Proposition 2.2. For the last estimate, we use the fact that $\sum_{i=1}^N \gamma_i(t) \varphi_i$ is bounded in $L^2((0, T_0); H_0^1(\Omega))$ and $(Y_{S,p}^N)$ is bounded in $W^{1,\infty}(0, T_0; H^3(\Omega))^3$. \square

For $R_{2,i}^N$ (we use the same argument for $R_{1,i}^N$), integrating by parts in the variable s , we have

$$\begin{aligned} R_{2,i}^N &= \int_0^{T_0-h} \int_{\Omega} \int_t^{t+h} (\rho_F^N u^N)(s, x) \partial_s w_i^N(t, s, x) ds dx dt \\ &\quad - \int_0^{T_0-h} \int_{\Omega} (\rho_F^N u^N)(t+h, x) \left(\int_t^{t+h} \partial_s w_i^N(t, s, x) ds \right) dx dt. \end{aligned}$$

And thus, according to (6.6), we obtain a uniform estimate on $R_{2,i}^N$

$$\begin{aligned} |R_{2,i}^N| &\leq C\sqrt{h} \|\rho_F^N u^N\|_{L^2((0,T)\times\Omega)} \left(\int_0^{T_0-h} \int_{\Omega} \int_t^{t+h} |\partial_s w_i^N(t, s, x)|^2 ds dx dt \right)^{\frac{1}{2}} \\ &\leq Ch. \end{aligned}$$

In order to estimate F_i^N , we consider on the interval $[t, t+h]$ the variational formulation (5.15) satisfied by our approximate solution with the test functions $w_i^N(t+h, s, x) - w_i^N(t, s, x)$. We treat separately F_1^N and F_2^N . For the estimate on F_1^N , we notice that

$$w_1^N(t+h, s, x) - w_1^N(t, s, x) = (V^N(t, s, X_S^N(0, t, x)))_p,$$

with

$$\begin{aligned} V^N(t, s, y) &= (\dot{a}^N(t+h) - \dot{a}^N(t)) + (\omega^N(t+h) - \omega^N(t)) \wedge (Q^N(s)(y - g_0 + \xi^N(s))) \\ &\quad + Q^N(s) V_E^N(t, s, y), \end{aligned} \tag{6.9}$$

and

$$V_E^N(t, s, y) = \sum_{i=1}^N (\alpha'_i(t+h) - \alpha'_i(t)) \psi_i(y) + (\beta_0(t+h, s) - \beta_0(t, s)) \psi_0(y).$$

The variational formulation gives

$$\begin{aligned} &\int_t^{t+h} \int_{\Omega_S(0)} \rho_S^0(y) \partial_s^2 X_S^N(s, 0, y) V^N(t, s, y) dy ds \\ &+ \epsilon \int_t^{t+h} (\partial_s^2 \xi^N, V_E^N)_{H^3(\Omega_S(0))} ds + \int_t^{t+h} \int_{\Omega_S(0)} \Sigma_E^2(\xi^N) : \varepsilon_y(V_E^N) dy ds \\ &+ \int_t^{t+h} \int_{\Omega} \partial_s (\rho_F^N u^N) \cdot (w_1^N(t+h, s, x) - w_1^N(t, s, x)) dx ds \end{aligned}$$

$$\begin{aligned}
& + \int_t^{t+h} \int_{\Omega} \operatorname{div}(\rho_E^N(u^N \otimes u^N)) \cdot (w_1^N(t+h, s, x) - w_1^N(t, s, x)) \, dx \, ds \\
& + 2\nu \int_t^{t+h} \int_{\Omega_F^N(s)} \varepsilon_x(u^N) : \varepsilon_x(w_1^N(t+h, s, x) - w_1^N(t, s, x)) \, dx \, ds = 0.
\end{aligned}$$

With the expression (6.5) of β_0 , it is not difficult to show that β_0 is bounded in $L_t^\infty(0, T; W_s^{1, \infty}(0, T))$. Thus we can write V_E^N as:

$$V_E^N(t, s, y) = (\partial_t \xi^N(t+h, y) - \partial_t \xi^N(t, y)) + W_E^N(t, s, y),$$

where W_E^N is bounded in $L_t^\infty(0, T; W_s^{1, \infty}(0, T; L^2(\Omega_S(0))))^3$. We can also write V^N as:

$$V^N(t, s, y) = (\partial_t X^N(t+h, 0, y) - \partial_t X^N(t, 0, y)) + W^N(t, s, y),$$

where W^N is bounded in $L_t^\infty(0, T; W_s^{1, \infty}(0, T; L^2(\Omega_S(0))))^3$. Thus, thanks to integrations by parts in the variable s and thanks to energy estimate (5.16), the first term of the variational formulation can be estimated by:

$$\begin{aligned}
& \left| \int_0^{T_0-h} \int_t^{t+h} \int_{\Omega_S(0)} \rho_S^0(y) \partial_s^2 X_S^N(s, 0, y) V^N(t, s, y) \, dy \, ds \, dt \right. \\
& \quad \left. - \int_0^{T_0-h} \int_{\Omega_S(0)} \rho_S^0(y) |\partial_t X^N(t+h, 0, y) - \partial_t X^N(t, 0, y)|^2 \, dy \, dt \right| \\
& \leq Ch \|W^N\|_{L_t^\infty(0, T; W_s^{1, \infty}(0, T; L^2(\Omega_S(0))))^3} \leq Ch,
\end{aligned}$$

where C depends on E_0 , ϵ and the data. So, with the same argument as above,

$$\begin{aligned}
& \left| \int_0^{T_0-h} \int_t^{t+h} (\partial_s^2 \xi^N, V_E^N)_{H^3(\Omega_S(0))} \, ds \, dt \right. \\
& \quad \left. - \int_0^{T_0-h} \|\partial_t \xi^N(t+h, y) - \partial_t \xi^N(t, y)\|_{H^3(\Omega_S(0))}^2 \, dt \right| \\
& \leq Ch \|W_E^N\|_{L_t^\infty(0, T; W_s^{1, \infty}(0, T; L^2(\Omega_S(0))))^3}
\end{aligned}$$

It follows that

$$\begin{aligned}
& |F_1^N| + \int_0^{T_0-h} \int_{\Omega_S(0)} \rho_S^0(y) |\partial_t X^N(t+h, 0, y) - \partial_t X^N(t, 0, y)|^2 \, dy \, dt \\
& \quad + \epsilon \int_0^{T_0-h} \|\partial_t \xi^N(t+h, y) - \partial_t \xi^N(t, y)\|_{H^3(\Omega_S(0))}^2 \, dt \\
& \leq Ch + \left| \int_0^{T_0-h} \int_t^{t+h} \int_{\Omega_S(0)} \Sigma_E^2(\xi^N) : \varepsilon_y(V_E^N) \, dy \, ds \, dt \right|
\end{aligned}$$

$$\begin{aligned}
& + \left| \int_0^{T_0-h} \int_t^{t+h} \int_{\Omega} \operatorname{div}(\rho_F^N u^N \otimes u^N)(s, x) (w_1^N(t+h, s, x) - w_1^N(t, s, x)) \, dx \, ds \, dt \right| \\
& + \left| 2\nu \int_0^{T_0-h} \int_t^{t+h} \int_{\Omega_F^N(s)} \varepsilon_x(u^N(s, x)) : \varepsilon_x(w_1^N(t+h, s, x) - w_1^N(t, s, x)) \, dx \, ds \, dt \right| \\
& \leq Ch + Ch^{\frac{2}{3}} \left(\int_0^{T_0-h} \int_{\Omega} \int_t^{t+h} |\nabla w_1^N(t, s, x)|^3 \, ds \, dx \, dt \right)^{\frac{1}{3}},
\end{aligned}$$

since (ξ^N) is bounded in $W^{1,\infty}(0, T; H^3(\Omega_S(0)))$, and (u^N) is bounded in $L^3((0, T) \times \Omega)$. By this way, thanks to (6.7), we have:

$$\begin{aligned}
& |F_1^N| + \int_0^{T_0-h} \int_{\Omega_S(0)} \rho_S^0(y) |\partial_t X^N(t+h, 0, y) - \partial_t X^N(t, 0, y)|^2 \, dy \, dt \\
& + \epsilon \int_0^{T_0-h} \|\partial_t \xi^N(t+h, y) - \partial_t \xi^N(t, y)\|_{H^3(\Omega_S(0))}^2 \, dt \leq Ch.
\end{aligned}$$

Finally,

$$\begin{aligned}
F_2^N &= \int_0^{T_0-h} \int_t^{t+h} \int_{\Omega} (\rho_F^N u^N \otimes u^N)(s, x) : \nabla(w_2^N(t+h, s, x) - w_2^N(t, s, x)) \, dx \, ds \, dt \\
& - 2\nu \int_0^{T_0-h} \int_t^{t+h} \int_{\Omega_F^N(s)} \varepsilon_x(u^N(s, x)) : \varepsilon_x(w_2^N(t+h, s, x) - w_2^N(t, s, x)) \, dx \, ds \, dt.
\end{aligned}$$

As (u^N) is bounded in $L^2(0, T_0; L^6(\Omega))$ and in $L^4(0, T_0; L^3(\Omega))$,

$$|F_2^N| \leq Ch^{\frac{3}{4}} \int_0^{T_0-h} \left(\int_t^{t+h} \left(\int_{\Omega} |\nabla w_2^N(t, s, x)|^2 \, dx \right)^2 \, ds \right)^{\frac{1}{2}} \, dt \leq Ch^{\frac{3}{4}},$$

according to (6.8). Thus, reassembling all these inequalities, we have shown Lemma 6.3 with $\alpha = \frac{3}{4}$. We have also shown the estimate

$$\begin{aligned}
& \int_0^{T_0-h} \int_{\Omega} \left| \sqrt{\rho_F^N(t+h)} u^N(t+h) - \sqrt{\rho_F^N(t)} u^N(t) \right|^2 \, dx \, dt, \\
& + \int_0^{T_0-h} \rho_S^0(y) |\partial_t X_S^N(t+h, 0, y) - \partial_t X_S^N(t, 0, y)|^2 \, dx \, dt, \quad (6.10) \\
& + \epsilon \int_0^{T_0-h} \|\partial_t \xi^N(t+h, y) - \partial_t \xi^N(t, y)\|_{H^3(\Omega_S(0))}^2 \, dt \leq g(h),
\end{aligned}$$

with

$$\lim_{h \rightarrow 0} g(h) = 0.$$

This inequality allows us to assert the following result:

Theorem 6.5. *On the subsequences already extracted in Proposition 5.2, we have the following strong convergences:*

$$\begin{aligned} \sqrt{\rho_F^N} u^N &\rightarrow \sqrt{\rho_F} u \text{ in } L^2((0, T_0) \times \Omega), \\ \xi^N &\rightarrow \xi \text{ in } H^1(0, T_0; H^s(\Omega_S(0))), \forall s < 3, \\ \omega^N &\rightarrow \omega \text{ in } L^2(0, T_0)^3, \\ a^N &\rightarrow a \text{ in } H^2(0, T_0)^3, \\ X_S^N &\rightarrow X_S \text{ in } H^1(0, T_0; H^s(\Omega_S(0))), \forall s < 3. \end{aligned}$$

Proof. From (6.10), we deduce that:

$$\begin{aligned} &\int_0^{T_0-h} \int_{\Omega_S(0)} |\dot{a}^N(t+h) - \dot{a}^N(t) + (\omega^N(t+h) - \omega^N(t)) \wedge (Q^N(t+h)(y - g_0 + \xi^N(t+h))) \\ &+ Q^N(t+h)(\partial_t \xi^N(t+h, y) - \partial_t \xi^N(t, y))|^2 dy dt \leq Ch. \end{aligned}$$

Thus, as $\xi^N(t)$ belongs to \mathcal{Y}_δ , for all $t \in [0, T_0]$, we obtain separately the following estimates: $\forall N \in \mathbb{N}$,

$$\begin{aligned} \int_0^{T_0-h} |\dot{a}^N(t+h) - \dot{a}^N(t)|^2 dt &\leq Ch, \quad \int_0^{T_0-h} |\omega^N(t+h) - \omega^N(t)|^2 dt \leq Ch, \\ \int_0^{T_0-h} \int_{\Omega_S(0)} |\partial_t \xi^N(t+h, y) - \partial_t \xi^N(t, y)|^2 dy dt &\leq Ch. \end{aligned}$$

We conclude with Lemma 6.1. \square

7. Conclusion

7.1. Passage to the limit

We do not give here all the details for this step. A complete explanation is given in [5]. We pass to the continuous problem thanks to the compactness results proven in Theorem 6.5.

In the variational formulation, we have to be careful while passing to the limit because test functions depend on N . Thus we consider test functions of the type

$$v^{M,N}(t, x) = (V^{M,N}(t, X_S^N(0, t, x)))_p + \sum_{i=1}^M C_i(t) [(Y_{S,p}^N(0, t, \cdot))^* \varphi_i](t, x),$$

with

$$V^{M,N}(t, y) = \dot{b}^M(t) + r^M(t) \wedge (Q^N(t)(y - g_0 + \xi^N(t, y))) + Q^N(t) V_E^{M,N}(t, y)$$

and

$$V_E^{M,N}(t, y) = \sum_{i=1}^M B_i(t) \psi_i(y) + B_0^{M,N}(t) \psi_0(y),$$

with $b^M \in H^2(0, T_0)^3$, $r^M \in H^1(0, T_0)^3$, $B_i \in H^1(0, T_0)$, $C_i \in H^2(0, T_0)$, $1 \leq i \leq M$ and with $B_0^{M,N}$ such that

$$\int_{\partial\Omega_S^N(t)} V_E^{M,N}(t, X_S^N(0, t, x)) \cdot n_x \, d\gamma_x = 0.$$

We first fix M and we consider $N \geq M$. For the strong convergence of test functions when N goes to $+\infty$, we need the following lemma:

Lemma 7.1. *Let $w^N(t)$ be defined on $\Omega_S^N(t) = X^N(t, 0, \Omega_S(0))$, for each $t \in [0, T_0]$, for each $N \in \mathbb{N}$. We extend, for every $N \in \mathbb{N}$ and for every $t \in [0, T_0]$, $w^N(t)$ to Ω by $w_p^N(t)$ defined on $\Omega_F^N(t)$ as the solution of*

$$\begin{cases} -\Delta w_p^N + \nabla p^N = 0 & \text{on } \Omega_F^N(t), \\ \operatorname{div} w_p^N = 0 & \text{on } \Omega_F^N(t), \\ w_p^N = w^N & \text{on } \partial\Omega_S^N(t), \\ w_p^N = 0 & \text{on } \partial\Omega \cap \partial\Omega_F^N(t). \end{cases}$$

We suppose that there exist $X \in H^1(0, T_0; H^3(\Omega_S(0)))^3$ and $w \in H^1(0, T_0; H^3(\Omega_S(0)))^3$ such that, for all $0 < s < 3$

$$\begin{aligned} X^N &\rightarrow X \text{ in } H^1(0, T_0; H^s(\Omega_S(0)))^3, \\ w^N(t, X^N(t, 0, \cdot)) &\rightarrow w(t, X(t, 0, \cdot)) \text{ in } H^1(0, T_0; H^s(\Omega_S(0)))^3. \end{aligned}$$

Then (∇w_p^N) converges to ∇w_p in $L^2((0, T_0) \times \Omega)^3$ and $(\partial_t w_p^N)$ converges to $\partial_t w_p$ in $L^2((0, T_0) \times \Omega)^3$.

To show this lemma, thanks to a change of variables we write the Stokes problem solved by w_p^N on $\Omega_F^N(t)$ which is the limit set of $\Omega_F^N(t)$.

After this step, we can pass to the limit in M without any difficulty. Thus, we now have a weak solution of our problem in the sense of Definition 3.1. This solution is defined on $[0, T_0]$.

At last, this solution satisfies energy estimate (3.6) according to Proposition 5.2, energy estimate in finite dimension (5.16) and the fact that

$$\lim_{N \rightarrow \infty} E_0^N = E^0.$$

7.2. Extension of our solution

For the moment, we have proven the existence of a solution defined on $[0, T_0]$ with T_0 depending on ϵ , E_0 , κ_0 , κ_1 and κ_2 . We want to extend this definition interval. At time $t = T_0$, as $T_0 \leq T$ with T defined by condition (3.7), we can

now start from the new configuration reference $\Omega_S(T_0)$ and we look for a solution $(\bar{\rho}_F, \bar{u})$. For the initial conditions, we have to bring back U_E and ρ_S to $\Omega_S(T_0)$. The representation of velocities has to be done again. To do this, it is necessary to reconstruct a function $\bar{\eta}$ which balance volume variations. But the unit outward normal \bar{n} is now a function of $H^{\frac{3}{2}}(\partial\Omega_S(0))$. Indeed, it is defined by (5.10) with $t = T_0$ and with X instead of \tilde{X}^N . Thus, we choose $\bar{\eta} \in H^3(\Omega_S(T_0))^3$ such that

$$\int_{\Omega_S(T_0)} \rho_S(T_0) \bar{\eta} dy = 0, \quad \int_{\Omega_S(T_0)} \rho_S(T_0) \bar{\eta} \wedge (y - a(T_0)) dy = 0, \quad (7.1)$$

$$\|\bar{n} - \bar{\eta}\|_{C(\partial\Omega_S(T_0))} \leq \frac{1}{4}. \quad (7.2)$$

For the proof of Lemma 4.1, we follow the proof of the implicit function theorem on the function \bar{f} where we replace η in the definition of f by $\bar{\eta}$ and $\Omega_S(0)$ by $\Omega_S(T_0)$. We notice that

$$\begin{aligned} \frac{\partial \bar{f}}{\partial \lambda}(0, 0) &= \int_{\partial\Omega_S(T_0)} \bar{\eta}(y) \cdot \bar{n}(y) d\gamma(y) \\ &\geq \frac{3}{4} \int_{\partial\Omega_S(T_0)} d\gamma(y) \geq \frac{3}{4} a(\partial\Omega_S(T_0)) \geq \frac{3}{4} a(\partial B), \end{aligned}$$

where B is the ball which has the same volume as $\Omega_S(T_0)$. Thanks to the volume conservation, B is independent of T_0 . By this way, we can prove that ρ_2 defined by Lemma 4.1 can be chosen independently of T_0 . As on $\Omega_S(0)$, we define on $\Omega_S(T_0)$ an extension operator $\bar{\mathcal{P}}$. The continuity constant $C_{\bar{\mathcal{P}}}$ depends only on the norm of $X_S(T_0, 0, \cdot)$ in $H^3(\Omega_S(0))$. Thus, thanks to energy estimate (3.6), $\bar{\kappa}_0$, $\bar{\kappa}_1$ and $\bar{\kappa}_2$ are independent of T_0 . So, as the existence time depends on ϵ , E_0 , $\bar{\kappa}_0$, $\bar{\kappa}_1$ and $\bar{\kappa}_2$, we have an existence time depending only on ϵ , E_0 , the data m , $\Omega_S(0)$ and on the constants δ , d_0 and γ_0 appearing in (3.7). By this way, we obtain a solution defined on $[0, T]$ with T defined in Theorem 3.4. This finishes the proof of Theorem 3.4.

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