Boulakia et al. (2008) "A Coupled System of PDEs and ODEs Arising in Electrocardiograms Modeling,"
Applied Mathematics Research eXpress, Vol. 2008, Article ID abn002, 28 pages.
doi:10.1093/amrx/abn002

A Coupled System of PDEs and ODEs Arising in Electrocardiograms Modeling

Muriel Boulakia[1], Miguel Angel Fernández[2], Jean-Frédéric Gerbeau[2], Nejib Zemzemi[2,3]

[1]Université Paris 6, Laboratoire Jacques-Louis Lions, REO project-team, F-75005 Paris, France, [2]INRIA, REO project-team, Rocquencourt, BP 105, F–78153 Le Chesnay Cedex, France and [3]Université Paris 11, Laboratoire de mathématiques d'Orsay, Bâtiment 425, 91405 Orsay Cedex, France

Correspondence to be sent to: jean-frederic.gerbeau@inria.fr

We study the well-posedness of a coupled system of PDEs and ODEs arising in the numerical simulation of electrocardiograms. It consists of a system of degenerate reaction–diffusion equations, the so-called bidomain equations, governing the electrical activity of the heart, and a diffusion equation governing the potential in the surrounding tissues. Global existence of weak solutions is proved for an abstract class of ionic models including Mitchell–Schaeffer, FitzHugh–Nagumo, Aliev–Panfilov, and McCulloch. Uniqueness is proved in the case of the FitzHugh–Nagumo ionic model. The proof is based on a regularization argument with a Faedo–Galerkin/compactness procedure.

1 Introduction

We analyze the well-posedness of a coupled system arising in the numerical simulation of electrocardiograms (ECG). It consists of two partial differential equations (PDEs) and a system of ordinary differential equations (ODEs), describing the electrical activity of the

Received August 17, 2007; Accepted October 16, 2007

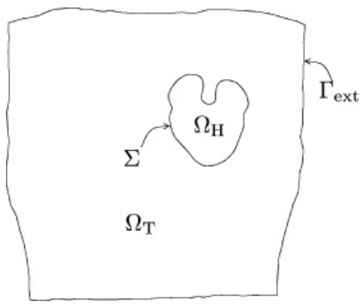

Fig. 1. The heart and torso domains: $\Omega_{\rm H}$ and $\Omega_{\rm T}$.

heart, coupled to a third PDE that describes the electrical potential of the surrounding tissue within the torso.

We assume the cardiac tissue to be located in a domain (an open bounded subset with locally Lipschitz continuous boundary) $\Omega_{\rm H}$ of $\mathbb{R}^3$. The surrounding tissue within the torso occupies a domain $\Omega_{\rm T}$. We denote by $\Sigma \stackrel{\rm def}{=} \overline{\Omega_{\rm H}} \cap \overline{\Omega_{\rm T}} = \partial\Omega_{\rm H}$ the interface between both domains, and by $\Gamma_{\rm ext}$ the external boundary of $\Omega_{\rm T}$, i.e. $\Gamma_{\rm ext} \stackrel{\rm def}{=} \partial\Omega_{\rm T}\backslash\Sigma$, see Figure 1. At last, we define Ω the global domain $\overline{\Omega_{\rm H}} \cup \Omega_{\rm T}$.

A widely accepted model of the macroscopic electrical activity of the heart is the so-called *bidomain model* (see, e.g. the monographs [20, 23, 24]). It consists of two degenerate parabolic reaction–diffusion PDEs coupled to a system of ODEs:

$$\begin{cases} C_{\rm m}\partial_t v_{\rm m} + I_{\rm ion}(v_{\rm m}, w) - \operatorname{div}(\boldsymbol{\sigma}_{\rm i}\nabla u_{\rm i}) = I_{\rm app}, & \text{in } \Omega_{\rm H} \times (0, T), \\ C_{\rm m}\partial_t v_{\rm m} + I_{\rm ion}(v_{\rm m}, w) + \operatorname{div}(\boldsymbol{\sigma}_{\rm e}\nabla u_{\rm e}) = I_{\rm app}, & \text{in } \Omega_{\rm H} \times (0, T), \\ \partial_t w + g(v_{\rm m}, w) = 0, & \text{in } \Omega_{\rm H} \times (0, T). \end{cases} \tag{1.1}$$

The two PDEs describe the dynamics of the averaged intra- and extracellular potentials $u_{\rm i}$ and $u_{\rm e}$, whereas the ODE, also known as *ionic model*, is related to the electrical behavior of the myocardium cells membrane, in terms of the (vector) variable w representing the averaged ion concentrations and gating states. In (1.1), the quantity $v_{\rm m} \stackrel{\rm def}{=} u_{\rm i} - u_{\rm e}$ stands for the transmembrane potential, $C_{\rm m}$ is the membrane capacitance, $\boldsymbol{\sigma}_{\rm i}, \boldsymbol{\sigma}_{\rm e}$ are the intra- and extracellular conductivity tensors and $I_{\rm app}$ is an external applied volume current. The nonlinear reaction term $I_{\rm ion}(v_{\rm m}, w)$ and the vector-valued function $g(v_{\rm m}, w)$ depend on the ionic model under consideration (e.g. Mitchell–Schaeffer [16], FitzHugh–Nagumo [17], or Luo–Rudy [14, 15]).

The PDE part of (1.1) has to be completed with boundary conditions for u_i and u_e. The intracellular domain is assumed to be electrically isolated, so we prescribe

$$\boldsymbol{\sigma}_i \nabla u_i \cdot n = 0, \quad \text{on } \Sigma,$$

where n stands for the outward unit normal on Σ. Conversely, the boundary conditions for u_e will depend on the interaction with the surrounding tissue.

The numerical simulation of the ECG signals requires a description of how the surface potential is perturbed by the electrical activity of the heart. In general, such a description is based on the coupling of (1.1) with a diffusion equation in Ω_T:

$$\operatorname{div}(\boldsymbol{\sigma}_T \nabla u_T) = 0, \quad \text{in } \Omega_T, \tag{1.2}$$

where u_T stands for the torso potential and $\boldsymbol{\sigma}_T$ for the conductivity tensor of the torso tissue. The boundary Γ_{ext} can be supposed to be insulated, which corresponds to the condition

$$\boldsymbol{\sigma}_T \nabla u_T \cdot n_T = 0 \quad \text{on } \Gamma_{ext},$$

where n_T stands for the outward unit normal on Γ_{ext}.

The coupling between (1.1) and (1.2) is operated at the heart–torso interface Σ. Generally, by enforcing the continuity of potentials and currents (see *e.g* [11, 13, 19, 20, 24]):

$$\begin{cases} u_e = u_T, & \text{on } \Sigma, \\ \boldsymbol{\sigma}_e \nabla u_e \cdot n = \boldsymbol{\sigma}_T \nabla u_T \cdot n, & \text{on } \Sigma. \end{cases} \tag{1.3}$$

These conditions represent a perfect electrical coupling between the heart and the surrounding tissue. More general coupling conditions, which take into account the impact of the pericardium (a double-walled sac that separates the heart and the surrounding tissue), have been reported by the authors in a recent work [4].

In summary, from (1.1), (1.2), and (1.3) we obtain the following coupled heart–torso model (see, e.g [11, 19, 20, 24]):

$$
\begin{cases}
C_{\rm m}\partial_t v_{\rm m} + I_{\rm ion}(v_{\rm m}, w) - \operatorname{div}(\boldsymbol{\sigma}_{\rm i}\nabla u_{\rm i}) = I_{\rm app}, & \text{in } \Omega_{\rm H},\\
C_{\rm m}\partial_t v_{\rm m} + I_{\rm ion}(v_{\rm m}, w) + \operatorname{div}(\boldsymbol{\sigma}_{\rm e}\nabla u_{\rm e}) = I_{\rm app}, & \text{in } \Omega_{\rm H},\\
\partial_t w + g(v_{\rm m}, w) = 0, & \text{in } \Omega_{\rm H},\\
\operatorname{div}(\boldsymbol{\sigma}_{\rm T}\nabla u_{\rm T}) = 0, & \text{in } \Omega_{\rm T},\\
\boldsymbol{\sigma}_{\rm i}\nabla u_{\rm i}\cdot n = 0, & \text{on } \Sigma\\
\boldsymbol{\sigma}_{\rm e}\nabla u_{\rm e}\cdot n = \boldsymbol{\sigma}_{\rm T}\nabla u_{\rm T}\cdot n, & \text{on } \Sigma,\\
u_{\rm e} = u_{\rm T}, & \text{on } \Sigma,\\
\boldsymbol{\sigma}_{\rm T}\nabla u_{\rm T}\cdot n_{\rm T} = 0, & \text{on} \Gamma_{\rm ext}.
\end{cases} \tag{1.4}
$$

Problem (1.4) is completed with initial conditions:

$$
v_{\rm m}(0, x) = v_0(x) \quad \text{and} \quad w(0, x) = w_0(x) \quad \forall\, x \in \Omega_{\rm H}, \tag{1.5}
$$

and the identity

$$
v_{\rm m} \stackrel{\rm def}{=} u_{\rm i} - u_{\rm e}, \quad \text{in } \Omega_{\rm H}. \tag{1.6}
$$

Finally, let us notice that $u_{\rm e}$ and $u_{\rm T}$ are defined up to the same constant. This constant can be fixed, for instance, by enforcing the following condition:

$$
\int_{\Omega_{\rm H}} u_{\rm e} = 0,
$$

on the extracellular potential.

Introduced in the late 70's [25], the system of Equations (1.1) can be derived mathematically using homogenization techniques. Typically, by assuming that the myocardium has periodic structure at the cell scale [12] (see also [7, 18]). A first well-posedness analysis of (1.1), with $I_{\rm ion}(v_{\rm m}, w)$ and $g(v_{\rm m}, w)$ given by the FitzHugh–Nagumo ionic model [17], has been reported in [7]. The proof is based on a reformulation of (1.1) in terms of an abstract evolutionary variational inequality. The analysis for a simplified ionic model, namely $I_{\rm ion}(v_{\rm m}, w) \stackrel{\rm def}{=} I_{\rm ion}(v_{\rm m})$, has been addressed in [2]. In the recent work [5], existence, uniqueness and regularity of a local, in time, solution are proved for the bidomain model with a general ionic model, using a semi-group approach. Existence of a global, in time, solution of the bidomain problem is also proved in [5] for a wide class of ionic models (including FitzHugh–Nagumo, Aliev–Panfilov [1], and

McCulloch [22]) through a compactness argument. Uniqueness, however, is achieved only for the FitzHugh–Nagumo ionic model. Finally, in [26], existence, uniqueness and some regularity results are proved with a generalized phase-I Luo–Rudy ionic model [14].

None of the above-mentioned works consider the coupled bidomain-torso problem (1.4). The aim of this paper is to provide a well-posedness analysis of this coupled problem. Our main result states the existence of global weak solutions for (1.4) with an abstract class of ionic models, including FitzHugh–Nagumo [10, 17], Aliev–Panfilov [1], Roger–McCulloch [22], and Mitchell–Schaeffer [16]. For the sake of completeness, we give here the expressions of I_{ion} and g for these models.

- FitzHugh–Nagumo model:

$$I_{\text{ion}}(v, w) = kv(v - a)(v - 1) + w, \quad g(v, w) = -\epsilon(\gamma v - w). \tag{1.7}$$

- Aliev–Panfilov model:

$$I_{\text{ion}}(v, w) = kv(v - a)(v - 1) + vw, \quad g(v, w) = \epsilon(\gamma v(v - 1 - a) + w). \tag{1.8}$$

- Roger–McCulloch model:

$$I_{\text{ion}}(v, w) = kv(v - a)(v - 1) + vw, \quad g(v, w) = -\epsilon(\gamma v - w). \tag{1.9}$$

- Mitchell–Schaeffer model:

$$\begin{aligned} I_{\text{ion}}(v, w) &= \frac{w}{\tau_{\text{in}}} v^2 (v - 1) - \frac{v}{\tau_{\text{out}}}, \\ g(v, w) &= \begin{cases} \dfrac{w - 1}{\tau_{\text{open}}} & \text{if } v \leq v_{\text{gate}}, \\ \dfrac{w}{\tau_{\text{close}}} & \text{if } v > v_{\text{gate}}. \end{cases} \end{aligned} \tag{1.10}$$

Here $0 < a < 1$, k, ϵ, γ, $\tau_{\text{in}} < \tau_{\text{out}} < \tau_{\text{open}}$, τ_{close} and $0 < v_{\text{gate}} < 1$ are given positive constants.

To the best of our knowledge, the ionic model (1.10) has not yet been considered within a well-posedness study of the bidomain equations (1.1). Compared to models (1.7)–(1.9), the Mitchell–Schaeffer model has different structure that makes the proof of our results slightly more involved. As far as the ECG modeling is concerned, in [3, 4], the authors point out that realistic ECG signals can be obtained with this model, whereas it seems to be not the case for standard FitzHugh–Nagumo type models (1.7).

The remainder of the paper is organized as follows. In the next section, we state our main existence result for problem (1.4), under general assumptions on the ionic model. In Section 3, we provide the proof of this result. We use a regularization argument and a standard Faedo–Galerkin/compactness procedure based on a specific spectral basis in Ω. Uniqueness is proved for the FitzHugh–Nagumo ionic model.

2 Main Result

We assume that the conductivities of the intracellular, extracellular, and thoracic media $\boldsymbol{\sigma}_{\rm i}$, $\boldsymbol{\sigma}_{\rm e}$, $\boldsymbol{\sigma}_{\rm T} \in [L^\infty(\Omega_{\rm H})]^{3\times 3}$ are symmetric and uniformly positive definite, i.e. there exist $\alpha_{\rm i} > 0$, $\alpha_{\rm e} > 0$, and $\alpha_{\rm T} > 0$ such that $\forall\, x \in \mathbb{R}^3$, $\forall\, \boldsymbol{\xi} \in \mathbb{R}^3$,

$$\boldsymbol{\xi}^T\boldsymbol{\sigma}_{\rm i}(x)\boldsymbol{\xi} \geq \alpha_{\rm i}|\boldsymbol{\xi}|^2, \quad \boldsymbol{\xi}^T\boldsymbol{\sigma}_{\rm e}(x)\boldsymbol{\xi} \geq \alpha_{\rm e}|\boldsymbol{\xi}|^2, \quad \boldsymbol{\xi}^T\boldsymbol{\sigma}_{\rm T}(x)\boldsymbol{\xi} \geq \alpha_{\rm T}|\boldsymbol{\xi}|^2. \tag{2.11}$$

Moreover, we shall use the notation $\alpha \stackrel{\rm def}{=} \min\{\alpha_{\rm e}, \alpha_{\rm T}\}$.

For the reaction terms we consider two kinds of (two-variable) ionic models:

- **I1**: Generalized FitzHugh–Nagumo models, where functions $I_{\rm ion}$ and g are given by

$$\begin{aligned} I_{\rm ion}(v,w) &= f_1(v) + f_2(v)w, \\ g(v,w) &= g_1(v) + c_1 w. \end{aligned} \tag{2.12}$$

  Here, f_1, f_2, and g_1 are given real functions and c_1 is a real constant.
- **I2**: A regularized version of the Mitchell–Schaeffer model (see, e.g. [9]), for which the functions $I_{\rm ion}$ and g are given by:

$$\begin{aligned} I_{\rm ion}(v,w) &= \frac{w}{\tau_{\rm in}} f_1(v) - \frac{v}{\tau_{\rm out}}, \\ g(v,w) &= \Big(\frac{1}{\tau_{\rm close}} + \frac{\tau_{\rm close} - \tau_{\rm open}}{\tau_{\rm close}\tau_{\rm open}} h_\infty(v)\Big)(w - h_\infty(v)), \end{aligned} \tag{2.13}$$

  where f_1 is a real function given by

$$f_1(v) = v^2(v-1), \tag{2.14}$$

the function h_∞ is given by

$$h_\infty(v) = \frac{1}{2}\left[1 - \tanh\left(\frac{v - v_{\text{gate}}}{\eta_{\text{gate}}}\right)\right], \tag{2.15}$$

and $\tau_{\text{in}}, \tau_{\text{out}}, \tau_{\text{open}}, \tau_{\text{close}}, v_{\text{gate}}, \eta_{\text{gate}}$ are positive constants.

In what follows we will consider the following two problems:

- **P1**: System (1.4) with the ionic model (**I1**) given by (2.12).
- **P2**: System (1.4) with the ionic model (**I2)** given by (2.13)-(2.15).

In order to analyze the well-posedness of these problems, we shall make use of the following assumptions on the behavior of the reaction terms.

- **A1**: We assume that f_1, f_2 and g_1 belong to $C^1(\mathbb{R})$ and that, $\forall\, v \in \mathbb{R}$,

$$\begin{aligned} |f_1(v)| &\le c_2 + c_3|v|^3, \\ f_2(v) &= c_4 + c_5 v, \\ |g_1(v)| &\le c_6 + c_7|v|^2, \end{aligned} \tag{2.16}$$

  with $\{c_i\}_{i=2}^{7}$ given real constants and c_2, c_3, c_6, c_7 are positives.
  For any $v \in \mathbb{R}$,

$$f_1(v)v \ge a|v|^4 - b|v|^2, \tag{2.17}$$

  with $a > 0$ and $b \ge 0$ given constants.
- **A2**: $(2.16)_1$ and (2.17).

The next assumption will be also used in order to prove uniqueness of the solution of problem **P1**.

- **A3**: For all $\mu > 0$, we introduce F_μ as

$$\begin{aligned} F_\mu : \mathbb{R}^2 &\longrightarrow \mathbb{R}^2 \\ (v, w) &\mapsto (\mu I_{\text{ion}}(v, w), g(v, w)), \end{aligned}$$

  and Q_μ as:

$$Q_\mu(z) \overset{\text{def}}{=} \frac{1}{2}\left(\nabla F_\mu(z) + \nabla F_\mu(z)^{\mathrm{T}}\right), \quad \forall\, z \in \mathbb{R}^2.$$

In addition, we assume that there exist $\mu_0 > 0$ and a constant $C_{\rm ion} \leq 0$ such that the eigenvalues $\lambda_{1,\mu_0}(z) \leq \lambda_{2,\mu_0}(z)$ of $Q_{\mu_0}(z)$, satisfy

$$C_{\rm ion} \leq \lambda_{1,\mu_0}(z) \leq \lambda_{2,\mu_0}(z), \quad \forall\, z \in \mathbb{R}^2. \tag{2.18}$$

Remark 2.1. One can check that models (1.7)–(1.9) enter the general framework (2.12) and satisfy the assumption **A1** and the model given by (2.13)–(2.15) satisfies assumption **A2**. In addition, **A3** holds true for the FitzHugh–Nagumo model. We refer to [5], for the details. □

In what follows, we shall make use of the following function spaces:

$$\begin{aligned}
V_{\rm i} &\stackrel{\rm def}{=} H^1(\Omega_{\rm H}),\\
V_{\rm e} &\stackrel{\rm def}{=} \left\{ \phi \in H^1(\Omega_{\rm H}) \,:\, \int_{\Omega_{\rm H}} \phi = 0 \right\},\\
V_{\rm HT} &\stackrel{\rm def}{=} \left\{ \phi \in H^1(\Omega_{\rm T}) \,:\, \phi_{|\Sigma} = 0 \right\},\\
V &\stackrel{\rm def}{=} \left\{ \phi \in H^1(\Omega) \,:\, \int_{\Omega_{\rm H}} \phi = 0 \right\}.
\end{aligned}$$

For times T, t and t_n we introduce the cylindrical time–space domains $Q_T \stackrel{\rm def}{=} (0,T) \times \Omega_{\rm H}$, $Q_t \stackrel{\rm def}{=} (0,t) \times \Omega_{\rm H}$, $Q_{t_n} \stackrel{\rm def}{=} (0,t_n) \times \Omega_{\rm H}$, and we define u as the extracellular cardiac potential in $\Omega_{\rm H}$, and the thoracic potential in $\Omega_{\rm T}$, *i.e.*:

$$u \stackrel{\rm def}{=} \begin{cases} u_{\rm e} & \text{in } \Omega_{\rm H},\\ u_{\rm T} & \text{in } \Omega_{\rm T}. \end{cases}$$

From the first coupling condition in (1.3), it follows that $u \in H^1(\Omega)$ provided that $u_{\rm e} \in H^1(\Omega_{\rm H})$ and $u_{\rm T} \in H^1(\Omega_{\rm T})$. Similarly, we define the global conductivity tensor $\boldsymbol{\sigma} \in [L^\infty(\Omega)]^{3\times 3}$ as

$$\boldsymbol{\sigma} \stackrel{\rm def}{=} \begin{cases} \boldsymbol{\sigma}_{\rm e} & \text{in } \Omega_{\rm H},\\ \boldsymbol{\sigma}_{\rm T} & \text{in } \Omega_{\rm T}. \end{cases}$$

Definition 2.1. A weak solution of problem **P1** is a quadruplet of functions (v_m, u_i, u, w) with the regularity

$$\begin{aligned} &v_m \in L^\infty(0,T;H^1(\Omega_H)) \cap H^1(0,T;L^2(\Omega_H)),\\ &u \in L^\infty(0,T;V), \quad w \in H^1(0,T;L^2(\Omega_H)), \end{aligned} \tag{2.19}$$

and satisfying (1.5), (1.6) and

$$C_m \int_{\Omega_H} \partial_t v_m \phi_i + \int_{\Omega_H} \boldsymbol{\sigma}_i \nabla u_i \cdot \nabla \phi_i + \int_{\Omega_H} I_{ion}(v_m, w)\phi_i = \int_{\Omega_H} I_{app}\phi_i, \tag{2.20}$$

$$C_m \int_{\Omega_H} \partial_t v_m \psi - \int_{\Omega} \boldsymbol{\sigma} \nabla u \cdot \nabla \psi + \int_{\Omega_H} I_{ion}(v_m, w)\psi = \int_{\Omega_H} I_{app}\psi, \tag{2.21}$$

$$\partial_t w + g(v_m, w) = 0. \tag{2.22}$$

for all $(\phi_i, \psi, \theta) \in H^1(\Omega_H) \times V \times L^2(\Omega_H)$. Equations (2.20) and (2.21) holds in $\mathcal{D}'(0,T)$ and Equation (2.22) holds almost everywhere. On the other hand, a weak solution of problem **P2** is a quadruplet (u_i, u, v_m, w) satisfying (2.19), (1.5), (1.6), (2.20)–(2.21) and

$$w \in W^{1,\infty}(0,T,L^\infty(\Omega_H)), \quad \partial_t w + g(v_m, w) = 0, \text{a.e. on } Q_T.$$

□

Remark 2.2. Since $w \in H^1(0,T;L^2(\Omega_H))$ it follows that $w \in C^0(0,T;L^2(\Omega_H))$, which gives a sense to the initial data of w. In the same manner, the initial condition on v_m makes sense. □

The next theorem provides the main result of this paper, it states the existence of solution for problems **P1** and **P2**.

Theorem 2.2. Let $T > 0$, $I_{app} \in L^2(Q_T)$, $\boldsymbol{\sigma}_i, \boldsymbol{\sigma}_e \in [L^\infty(\Omega_H)]^{3\times 3}$ symmetric and satisfying (2.11), $w_0 \in L^2(\Omega_H)$ and $v_0 \in H^1(\Omega_H)$ be given data.

- If **A1** holds, then problem **P1** has a weak solution in the sense of Definition 2.1. Moreover, if assumption **A3** holds true, the solution is unique.
- If **A2** holds and $w_0 \in L^\infty(\Omega_H)$ with a positive lower bound $r > 0$, such that

$$r < w_0 \leq 1 \quad \text{in } \Omega_H, \tag{2.23}$$

then, problem **P2** has a weak solution in the sense of Definition 2.1. □

The next section is fully devoted to the proof of this theorem.

3 Proof of the Main Result

Two main issues arise in the analysis of problem (1.4). First, the nonlinear reaction–diffusion equations $(1.4)_{1,2}$ are degenerate in time. And secondly, we have a coupling with a diffusion equation through the interface Σ. The first issue is overcome here by adding a couple of regularization terms, making bidomain equations parabolic. The method we propose simplifies the approach used in [2] by merging regularization and approximation of the solution . Then, the resulting regularized system can be analyzed by standard arguments, namely, through a Faedo–Galerkin/compactness procedure and a specific treatment of the nonlinear terms. On the other hand, the second matter can be handled through a specific definition of the Galerkin basis.

In paragraph 3.1, regularization and Faedo–Galerkin techniques are merged by introducing a regularized problem in finite dimension n. In the next paragraph, existence of solution for this problem is proved. In paragraph 3.3, energy estimates are derived, independent of the regularization parameter $\frac{1}{n}$. Existence of solution for the continuous problem is addressed in Section 3.4 whereas, in 3.5, uniqueness is proved for problem **P1**, under the additional assumption **A3**.

3.1 A regularized problem in finite dimension

Let $\{h_k\}_{k\in\mathbb{N}^*}$ be a Hilbert basis of $V_{\rm i}$, $\{f_k\}_{k\in\mathbb{N}^*}$ be a Hilbert basis of $V_{\rm e}$ and $\{g_k\}_{k\in\mathbb{N}^*}$ a Hilbert basis of $V_{\rm HT}$; see, e.g. [8]. We assume that these basis functions are (sufficiently) smooth and that $\{h_k\}_{k\in\mathbb{N}^*}$ is an orthogonal basis in $L^2(\Omega_{\rm H})$ (see, e.g. [21] page 268). We introduce a Galerkin basis of V by defining, for all $k\in\mathbb{N}^*$, $\tilde{f}_k \in H^1(\Omega)$ as an extension of f_k in $H^1(\Omega)$, given by an arbitrary continuous extension operator. We also extend, for all $k\in\mathbb{N}^*$, g_k by $\tilde{g}_k \in H^1(\Omega)$ such that $\tilde{g}_k = 0$ in $\Omega_{\rm H}$. One can check straightforwardly that $\{e_k\}_{k\in\mathbb{N}^*}$, defined as, $e_{2k-1} = \tilde{f}_k, \quad e_{2k} = \tilde{g}_k, \quad \forall\, k\in\mathbb{N}^*$, is a Galerkin basis of V.

Finally, for all $n\in\mathbb{N}^*$, we can define the finite-dimensional spaces $V_{{\rm i},n}$, $V_{{\rm e},n}$, $V_{{\rm T},n}$ and V_n generated, respectively, by $\{h_k\}_{k=1}^n$,$\{f_k\}_{k=1}^n$, $\{g_k\}_{k=1}^n$ and $\{e_k\}_{k=1}^{2n}$, *i.e*

$$
\begin{aligned}
&V_{{\rm i},n} \stackrel{\rm def}{=} < \{h_k\}_{k=1}^n >, \quad V_{{\rm e},n} \stackrel{\rm def}{=} < \{f_k\}_{k=1}^n >,\\
&V_{{\rm T},n} \stackrel{\rm def}{=} < \{g_k\}_{k=1}^n >, \quad V_n \stackrel{\rm def}{=} < \{e_k\}_{k=1}^{2n} > .
\end{aligned}
$$

Hence, we can introduce, for each $n\in\mathbb{N}^*$, the following two discrete problems $\mathbf{P1}_n$ and $\mathbf{P2}_n$ associated with problems **P1** and **P2**, respectively:

- **P1**$_n$: Find $(u_{i,n}, u_n) \in C^1(0,T; V_{i,n} \times V_n)$, $w_n \in C^1(0,T; V_{i,n})$ such that, for $v_n = u_{i,n} - u_{n/\Omega_H}$ and for all $(h, e, \theta) \in V_{i,n} \times V_n \times V_{i,n}$ we have,

$$
\begin{aligned}
&C_m \int_{\Omega_H} \partial_t v_n h + \frac{1}{n}\int_{\Omega_H} \partial_t u_{i,n} h + \int_{\Omega_H} \boldsymbol{\sigma}_i \nabla u_{i,n} \cdot \nabla h \\
&\qquad\qquad + \int_{\Omega_H} I_{ion}(v_n, w_n) h = \int_{\Omega_H} I_{app} h, \\
&C_m \int_{\Omega_H} \partial_t v_n e - \frac{1}{n}\int_{\Omega} \partial_t u_n e - \int_{\Omega} \boldsymbol{\sigma} \nabla u_n \cdot \nabla e \\
&\qquad\qquad + \int_{\Omega_H} I_{ion}(v_n, w_n) e = \int_{\Omega_H} I_{app} e, \\
&\int_{\Omega_H} \partial_t w_n \theta + \int_{\Omega_H} g(v_n, w_n)\theta = 0,
\end{aligned}
\tag{3.24}
$$

with $v_n \stackrel{\text{def}}{=} u_{i,n} - u_{n|\Omega_H}$ and verifying the initial conditions

$$
\begin{aligned}
&v_n(0) = v_{0,n}, \quad u_{i,n}(0) = u_{i,0,n}, \quad \text{in } \Omega_H; \\
&u_n(0) = u_{0,n} \quad \text{in } \Omega, \quad w_n(0) = w_{0,n}, \quad \text{in } \Omega_H,
\end{aligned}
\tag{3.25}
$$

Here, $v_{0,n}, w_{0,n}$ are suitable approximations of v_0 and w_0 in $V_{i,n}$, and $u_{i,0,n}, u_{0,n}$ are *auxiliary* initial conditions to be specified later on.

- **P2**$_n$: Find $(u_{i,n}, u_n) \in C^1(0,T; V_{i,n} \times V_n)$ and $w_n \in C^1(0,T, L^\infty(\Omega_H))$ such that, for $v_n = u_{i,n} - u_{n/\Omega_H}$, the triplet $(v_n, u_{i,n}, u_n)$ satisfy $(3.24)_{1,2}$-$(3.25)_1$ and

$$
\begin{aligned}
\partial_t w_n + g(v_n, w_n) = 0, &\quad \text{a.e. in} \quad Q_T, \\
w_n(0) = w_0, &\quad \text{a.e. in} \quad \Omega_H.
\end{aligned}
\tag{3.26}
$$

The (auxiliary) initial conditions for $u_{i,n}$ and u_n, needed by the two problems below, are defined by introducing two arbitrary functions $u_{i,0} \in H^1(\Omega_H)$ and $u_0 \in V$ such that $v_0 = u_{i,0} - u_0$ in Ω_H. Then, for $n \in \mathbb{N}^*$, we define $(u_{i,0,n}, u_{0,n}, w_{0,n})$ as the orthogonal projections, on $V_{i,n} \times V_n \times V_{i,n}$, of $(u_{i,0}, u_0, w_0)$. Clearly, by construction of these sequences, we have

$$
(v_{0,n}, u_{i,0,n}, u_{0,n}, w_{0,n}) \longrightarrow (v_0, u_{i,0}, u_0, w_0), \tag{3.27}
$$

in $V_i^2 \times V \times L^2(\Omega_H)$.

3.2 Local existence of the discretized solution

Lemma 3.1. Suppose that there exists C_0 such that

$$\|u_{\mathrm{i},0,n}\|_{H^1(\Omega_{\mathrm{H}})} + \|u_{0,n}\|_{H^1(\Omega)} + \|w_{0,n}\|_{L^2(\Omega_{\mathrm{H}})} \leq C_0. \tag{3.28}$$

For all $n \in \mathbb{N}^*$ there exists a positive time $0 < t_n \leq T$ which only depends on C_0 such that problems $\mathbf{P1}_n$ and $\mathbf{P2}_n$ admit a unique solution over the time interval $[0, t_n]$. □

Proof. For the sake of conciseness we only give here the details of the proof for problem $\mathbf{P1}_n$, the proof for problem $\mathbf{P2}_n$ follows with minor modifications. Since $\{h_l\}_{1\leq l\leq n}$ and $\{e_l\}_{1\leq l\leq 2n}$ are basis of $V_{\mathrm{i},n}$ and V_{n}, respectively, we can write

$$\begin{aligned} u_{\mathrm{i},n}(t) &= \sum_{l=1}^{n} c_{\mathrm{i},l}(t) h_l, \quad u_n(t) = \sum_{l=1}^{2n} c_l(t) e_l, \quad w_n(t) = \sum_{l=1}^{n} c_{\mathrm{w},l}(t) h_l, \\ u_{\mathrm{i},0,n} &= \sum_{l=1}^{n} c^0_{\mathrm{i},l} h_l, \quad u_{0,n} = \sum_{l=1}^{2n} c^0_l e_l, \quad w_{0,n} = \sum_{l=1}^{n} c^0_{\mathrm{w},l} h_l. \end{aligned} \tag{3.29}$$

Thus, introducing the notations

$$\begin{aligned} c_{\mathrm{i}} &\stackrel{\mathrm{def}}{=} \{c_{\mathrm{i},l}\}_{l=1}^{n}, \quad c \stackrel{\mathrm{def}}{=} \{c_l\}_{l=1}^{2n}, \quad c_{\mathrm{w}} \stackrel{\mathrm{def}}{=} \{c_{\mathrm{w},l}\}_{l=1}^{n}, \\ c^0_{\mathrm{i}} &\stackrel{\mathrm{def}}{=} \{c^0_{\mathrm{i},l}\}_{l=1}^{n}, \quad c^0 \stackrel{\mathrm{def}}{=} \{c^0_l\}_{l=1}^{2n}, \quad c^0_{\mathrm{w}} \stackrel{\mathrm{def}}{=} \{c^0_{\mathrm{w},l}\}_{l=1}^{n}, \end{aligned}$$

it follows that problem $\mathbf{P1}_n$ is equivalent to the following nonlinear system of ordinary differential equations (ODE)

$$\mathbf{M} \begin{bmatrix} c'_{\mathrm{i}} \\ c' \\ c'_{\mathrm{w}} \end{bmatrix} = \begin{bmatrix} G_{\mathrm{i}}\big(t, c_{\mathrm{i}}, c, c_{\mathrm{w}}\big) \\ G\big(t, c_{\mathrm{i}}, c, c_{\mathrm{w}}\big) \\ G_{\mathrm{w}}\big(t, c_{\mathrm{i}}, c, c_{\mathrm{w}}\big) \end{bmatrix}, \quad \begin{bmatrix} c_{\mathrm{i}}(0) \\ c(0) \\ c_{\mathrm{w}}(0) \end{bmatrix} = \begin{bmatrix} c^0_{\mathrm{i}} \\ c^0 \\ c^0_{\mathrm{w}} \end{bmatrix}. \tag{3.30}$$

Here, the *mass* matrix $M \in \mathbb{R}^{4n\times 4n}$ is given by

$$M \stackrel{\text{def}}{=} \left[\begin{array}{c:c:c} (C_\mathrm{m}+\frac{1}{n})M_{V_\mathrm{i}} & -C_\mathrm{m}M_{V_\mathrm{ie}} & 0 \\ \hdashline -C_\mathrm{m}{M_{V_\mathrm{ie}}}^T & C_\mathrm{m}M_{V_\mathrm{e}}+\frac{1}{n}M_{V_\mathrm{HT}} & 0 \\ \hdashline 0 & 0 & M_{V_\mathrm{i}} \end{array}\right],$$

with $M_{V_\mathrm{i}} \in \mathbb{R}^{n\times n}$, $M_{V_\mathrm{ie}} \in \mathbb{R}^{n\times 2n}$ and $M_{V_\mathrm{e}}, M_{V_\mathrm{HT}} \in \mathbb{R}^{2n\times 2n}$

$$M_{V_\mathrm{i}} \stackrel{\text{def}}{=} \left(\int_{\Omega_\mathrm{H}} h_k h_l\right)_{1\le k,l\le n}, \quad M_{V_\mathrm{ie}} \stackrel{\text{def}}{=} \left(\int_{\Omega_\mathrm{H}} h_k e_l\right)_{1\le k\le n,\,1\le l\le 2n},$$
$$M_{V_\mathrm{e}} \stackrel{\text{def}}{=} \left(\int_{\Omega_\mathrm{H}} e_k e_l\right)_{1\le k,l\le 2n}, \quad M_{V_\mathrm{HT}} \stackrel{\text{def}}{=} \left(\int_{\Omega} e_k e_l\right)_{1\le k,l\le 2n}.$$

On the other hand, from the notations

$$G_\mathrm{i} \stackrel{\text{def}}{=} \{G_{\mathrm{i},k}\}_{k=1}^{n}, \quad G \stackrel{\text{def}}{=} \{G_k\}_{k=1}^{2n}, \quad G_\mathrm{w} \stackrel{\text{def}}{=} \{G_{\mathrm{w},k}\}_{k=1}^{n},$$

the right-hand side of (3.30) is given by

$$G_{\mathrm{i},k}(t,c_\mathrm{i},c,c_\mathrm{w}) \stackrel{\text{def}}{=} -\int_{\Omega_H} \boldsymbol{\sigma}_\mathrm{i}\nabla u_{\mathrm{i},n}\cdot\nabla h_k - \int_{\Omega_H} I_\mathrm{ion}(v_n,w_n)h_k + \int_{\Omega_H} I_\mathrm{app}h_k,$$

for all $1 \le k \le n$,

$$G_k(t,c_\mathrm{i},c,c_\mathrm{w}) \stackrel{\text{def}}{=} -\int_{\Omega} \boldsymbol{\sigma}\nabla u_n\cdot\nabla e_k + \int_{\Omega_H} I_\mathrm{ion}(v_n,w_n)e_k - \int_{\Omega_H} I_\mathrm{app}e_k,$$

for all $1 \le k \le 2n$, and finally,

$$G_{\mathrm{w},k}(t,c_\mathrm{i},c,c_\mathrm{w}) \stackrel{\text{def}}{=} -\int_{\Omega_H} g(v_n,w_n)h_k,$$

for all $1 \le k \le n$.

According to Lemma 3.2, given below, the mass matrix M is positive definite and hence invertible and, on the other hand, the right-hand side of (3.30) is a C^1 function with respect to the arguments c_{i}, c, and c_{w}. Thus, thanks to Cauchy–Lipschitz theorem (we refer, for instance, to [6]), we obtain the existence of a local solution for the ODE system (3.30) defined on $[0, t_n]$ where t_n only depends on C_0 (introduced in (3.28)). This completes the proof. ■

Lemma 3.2. For all $n \in \mathbb{N}^*$, the matrix M is positive definite. □

Proof. We can decompose M as $M = C_{\mathrm{m}} N + \frac{1}{n} D$, with

$$D \stackrel{\text{def}}{=} \begin{bmatrix} M_{V_{\mathrm{i}}} & 0 & 0 \\ 0 & M_{V_{\mathrm{HT}}} & 0 \\ 0 & 0 & n M_{V_{\mathrm{i}}} \end{bmatrix},$$

and

$$N \stackrel{\text{def}}{=} \begin{bmatrix} M_{V_{\mathrm{i}}} & -M_{V_{\mathrm{ie}}} & 0 \\ -{M_{V_{\mathrm{ie}}}}^T & M_{V_{\mathrm{e}}} & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Since the matrices $M_{V_{\rm i}}$, $M_{V_{\rm HT}}$ and $M_{V_{\rm i}}$ are mass matrices, we obtain that the block-diagonal matrix D is positive definite. On the other hand, for each $\begin{bmatrix} c_{\rm i} & c & c_{\rm w}\end{bmatrix}^{\rm T} \in \mathbb{R}^{4n}$ we have

$$\begin{aligned}\begin{bmatrix} c_{\rm i} \\ c \\ c_{\rm w}\end{bmatrix}^{\rm T} N \begin{bmatrix} c_{\rm i} \\ c \\ c_{\rm w}\end{bmatrix} &= \sum_{l,k=1}^{n} \left(\int_{\Omega_{\rm H}} c_{{\rm i},l} c_{{\rm i},k} h_l h_k - 2 \int_{\Omega_{\rm H}} c_{{\rm i},l} c_{2k-1} h_l f_k + \int_{\Omega_{\rm H}} c_{2l-1} c_{2k-1} f_l f_k \right) \\ &= \left\| \sum_{l,1}^{n} \left(c_{{\rm i},l} h_l - c_{2l-1} f_l \right) \right\|^2_{L^2(\Omega_{\rm H})} \\ &\geq 0,\end{aligned}$$

so that N is positive. It then follows that M is positive definite. ■

Remark 3.1. The above lemma points out the role of the regularization term $\frac{1}{n}D$. It allows to obtain a matrix M in (3.30) which is nonsingular, so that the resulting system of ODE is nondegenerate. □

3.3 **Energy estimates**

In the next lemma, we state some uniform estimates (with respect to n) of the solution of problems $\mathbf{P1}_n$ and $\mathbf{P2}_n$. We also provide similar estimates for the time derivative, which will be useful for the passage to the limit. For the sake of clarity, in what follows, $c > 0$ stands for a generic constant, which depends on T, on the initial conditions, and on the physical parameters, but which is independent of n.

Lemma 3.3. Let $u_{{\rm i},0} \in H^1(\Omega_{\rm H})$, $u_0 \in V$, $w_0 \in L^2(\Omega_{\rm H})$ and $I_{\rm app} \in L^2(Q_T)$ be given data and let $(u_{{\rm i},n}, u_n, w_n)$ be a solution of $\mathbf{P1}_n$ defined on $[0, T']$ for $0 < T' < T$. Assume that **A1** holds true. Then, for $v_n = u_{{\rm i},n} - u_{n/\Omega_{\rm H}}$ and for all $n \in \mathbb{N}^*$ and $t \in [0, T']$, we have

$$\begin{aligned}&\|v_n\|_{L^\infty(0,t;L^2(\Omega_{\rm H}))} + \|v_n\|_{L^4(Q_t)} + \frac{1}{\sqrt{n}} \left(\|u_{{\rm i},n}\|_{L^\infty(0,t;L^2(\Omega_{\rm H}))} + \|u_n\|_{L^\infty(0,t;L^2(\Omega))} \right) \\ &\qquad + \|\nabla u_{{\rm i},n}\|_{L^2(Q_t)} + \|\nabla u_n\|_{L^2((0,t)\times\Omega)} \leq c, \\ &\|\partial_t v_n\|_{L^2(Q_t)} + \|v_n\|_{L^\infty(0,t;H^1(\Omega_{\rm H}))} + \frac{1}{\sqrt{n}} \left(\|\partial_t u_{{\rm i},n}\|_{L^2(Q_t)} + \|\partial_t u_n\|_{L^2((0,t)\times\Omega)} \right) \\ &\qquad + \|\nabla u_{{\rm i},n}\|_{L^\infty(0,t;L^2(\Omega_{\rm H}))} + \|\nabla u_n\|_{L^\infty(0,t;L^2(\Omega))} \leq c,\end{aligned} \tag{3.31}$$

and

$$\|w_n\|_{L^\infty(0,t;L^2(\Omega_{\rm H}))} \leq c, \quad \|\partial_t w_n\|_{L^2(Q_t)} \leq c. \tag{3.32}$$

If **A2** is satisfied and $w_0 \in L^\infty(\Omega_{\rm H})$ with (2.23), there exists a positive constant $w_{\min}$ (independent of T') such that a solution $(u_{{\rm i},n}, u_n, w_n)$ of $\mathbf{P2}_n$ defined on $[0, T']$ for $T' > 0$ satisfies (3.31) and, for all $t \in [0, T']$

$$\|w_n\|_{W^{1,\infty}(0,t,L^\infty(\Omega_{\rm H}))} \leq c, \quad w_{\min} \leq w_n \leq 1, \quad \text{in} Q_{T'}. \tag{3.33}$$

□

Proof. We start by proving the estimates for problem $\mathbf{P1}_n$. Taking $h = u_{{\rm i},n}$, $e = -u_n$, $\theta = w_n$ in (3.24) and using the uniform coercivity of the conductivity tensors (2.11), we obtain:

$$\begin{aligned}
&\frac{1}{2}\frac{\rm d}{\rm dt}\left[\|w_n\|^2_{L^2(\Omega_{\rm H})} + C_{\rm m}\|v_n\|^2_{L^2(\Omega_{\rm H})} + \frac{1}{n}\left(\|u_{{\rm i},n}\|^2_{L^2(\Omega_{\rm H})} + \|u_n\|^2_{L^2(\Omega)}\right)\right] \\
&\quad + \alpha_{\rm i}\|\nabla u_{{\rm i},n}\|^2_{L^2(\Omega_{\rm H})} + \alpha\|\nabla u_n\|^2_{L^2(\Omega)} + \int_{\Omega_{\rm H}} I_{\rm ion}(v_n, w_n)v_n \\
&\quad + \int_{\Omega_{\rm H}} g(v_n, w_n)w_n \leq \int_{\Omega_{\rm H}} I_{\rm app}v_n.
\end{aligned} \tag{3.34}$$

From assumption **A1**, we get

$$I_{\rm ion}(v, w)v + g(v, w)w \geq a|v|^4 - (c_8|v|^2 + c_9|w|^2) - c_{10},$$

with $c_8, c_9, c_{10} > 0$. Thus, inserting this expression in (3.34) and using the Cauchy–Schwarz's inequality, it follows that

$$\begin{aligned}
&\frac{1}{2}\frac{\rm d}{\rm dt}\left[\|w_n\|^2_{L^2(\Omega_{\rm H})} + C_{\rm m}\|v_n\|^2_{L^2(\Omega_{\rm H})} + \frac{1}{n}\left(\|u_{{\rm i},n}\|^2_{L^2(\Omega_{\rm H})} + \|u_n\|^2_{L^2(\Omega)}\right)\right] \\
&\quad + \alpha_{\rm i}\|\nabla u_{{\rm i},n}\|^2_{L^2(\Omega_{\rm H})} + \alpha\|\nabla u_n\|^2_{L^2(\Omega)} + a\|v_n\|^4_{L^4(\Omega_{\rm H})} \\
&\quad \leq \left(c_8 + \frac{1}{2}\right)\|v_n\|^2_{L^2(\Omega_{\rm H})} + c_9\|w_n\|^2_{L^2(\Omega_{\rm H})} + c_{10}|\Omega_{\rm H}| + \frac{1}{2}\|I_{\rm app}\|^2_{L^2(\Omega_{\rm H})}.
\end{aligned}$$

Therefore, integrating over $(0,t)$, with $t \in [0,T']$, we have

$$\begin{aligned}
&\|w_n\|^2_{L^2(\Omega_{\rm H})} + C_{\rm m}\|v_n\|^2_{L^2(\Omega_{\rm H})} + \frac{1}{n}\left(\|u_{{\rm i},n}\|^2_{L^2(\Omega_{\rm H})} + \|u_n\|^2_{L^2(\Omega)}\right)\\
&\quad + \alpha_{\rm i}\|\nabla u_{{\rm i},n}\|^2_{L^2(Q_t)} + \alpha\|\nabla u_n\|^2_{L^2(\Omega\times(0,t))} + a\|v_n\|^4_{L^4(Q_t)}\\
&\quad \le c\int_0^t \left(\|v_n\|^2_{L^2(\Omega_{\rm H})} + \|w_n\|^2_{L^2(\Omega_{\rm H})}\right) + c_{10}|\Omega_{\rm H}|T + \frac{1}{2}\|I_{\rm app}\|^2_{L^2(Q_T)}\\
&\quad + \|w_{0,n}\|^2_{L^2(\Omega_{\rm H})} + C_{\rm m}\|v_{0,n}\|^2_{L^2(\Omega_{\rm H})} + \frac{1}{n}\left(\|u_{{\rm i},0,n}\|^2_{L^2(\Omega_{\rm H})} + \|u_{0,n}\|^2_{L^2(\Omega)}\right),
\end{aligned}$$

for all $t \in [0,T']$. Estimates $(3.31)_1$ and $(3.32)_1$ follow by applying Gronwall lemma and using the fact that, from (3.27),

$$\|w_{0,n}\|^2_{L^2(\Omega_{\rm H})} + C_{\rm m}\|v_{0,n}\|^2_{L^2(\Omega_{\rm H})} + \frac{1}{n}\left(\|u_{{\rm i},0,n}\|^2_{L^2(\Omega_{\rm H})} + \|u_{0,n}\|^2_{L^2(\Omega)}\right),$$

is uniformly bounded with respect to n.

For the estimate of the time derivative, following [2], we notice that

$$\int_{\Omega_{\rm H}} f_1(v)\partial_t v = \frac{d}{dt}\int_{\Omega_{\rm H}} H(v), \quad H(v) \stackrel{\rm def}{=} \int_0^v f_1. \tag{3.35}$$

On the other hand, taking $h = \partial_t u_{{\rm i},n}$, $e = \partial_t u_n$ and $\theta = \partial_t w_n$ in (3.24) and integrating over $(0,t)$, with $t \in [0,T']$, yield

$$\begin{aligned}
&\|\partial_t w_n\|^2_{L^2(Q_t)} + C_{\rm m}\|\partial_t v_n\|^2_{L^2(Q_t)} + \frac{1}{n}\left(\|\partial_t u_{{\rm i},n}\|^2_{L^2(Q_t)} + \|\partial_t u_n\|^2_{L^2(0,t;L^2(\Omega))}\right)\\
&\quad + \frac{\alpha_{\rm i}}{2}\|\nabla u_{{\rm i},n}\|^2_{L^2(\Omega_{\rm H})} + \frac{\alpha}{2}\|\nabla u_n\|^2_{L^2(\Omega)}\\
&\quad \le \frac{1}{2}\int_{\Omega_{\rm H}} \boldsymbol{\sigma}_{\rm i}\nabla u_{{\rm i},0,n}\cdot\nabla u_{{\rm i},0,n} + \frac{1}{2}\int_{\Omega}\boldsymbol{\sigma}\nabla u_{0,n}\cdot\nabla u_{0,n} + \int_{\Omega_{\rm H}} H(v_{0,n})\\
&\quad - \int_{\Omega_{\rm H}} H(v_n) + \int_0^t\int_{\Omega_{\rm H}} I_{\rm app}\partial_t v_n - \int_0^t\int_{\Omega_{\rm H}} (f_2(v_n)w_n\partial_t v_n + g(v_n,w_n)\partial_t w_n).
\end{aligned} \tag{3.36}$$

It remains now to estimate the right-hand side of this expression. The first two terms can be bounded using (3.27). For the third term, we use $(2.16)_1$, the continuous embedding of $H^1(\Omega_{\rm H})$ into $L^4(\Omega_{\rm H})$ and (3.27) to obtain

$$\int_{\Omega_{\rm H}} \left|H(v_{0,n})\right| = \int_{\Omega_{\rm H}}\left|\int_0^{v_{0,n}} f_1(s)ds\right| \le \int_{\Omega_{\rm H}} c\left(v_{0,n}^4 + 1\right) \le c.$$

For the fourth term, according to assumption (2.17), we have $f_1(v)v + bv^2 \geq 0$. In other words, $f_1(v) + bv \geq 0$ for $v \geq 0$, and $f_1(v) + bv \leq 0$ for $v \leq 0$. As a result, integrating over $(0, v)$ yields

$$-H(v) \leq \frac{b}{2}v^2. \tag{3.37}$$

On the other hand, the fifth term can be controlled using the Cauchy–Schwarz inequality.

In summary, from (3.36) and (2.12), we get

$$\begin{aligned}
&\|\partial_t w_n\|^2_{L^2(Q_t)} + \frac{C_\mathrm{m}}{2}\|\partial_t v_n\|^2_{L^2(Q_t)} + \frac{1}{n}\|\partial_t u_{\mathrm{i},n}\|^2_{L^2(Q_t)} \\
&\quad + \frac{1}{n}\|\partial_t u_n\|^2_{L^2(0,t;L^2(\Omega))} + \frac{\alpha_\mathrm{i}}{2}\|\nabla u_{\mathrm{i},n}\|^2_{L^2(\Omega_\mathrm{H})} + \frac{\alpha}{2}\|\nabla u_n\|^2_{L^2(\Omega)} \\
&\quad \leq c + \frac{1}{2C_\mathrm{m}}\|I_\mathrm{app}\|^2_{L^2(Q_t)} + \frac{b}{2}\|v_n\|^2_{L^2(\Omega_\mathrm{H})} \\
&\quad - \int_0^t\int_{\Omega_\mathrm{H}} f_2(v_n)w_n\partial_t v_n - \int_0^t\int_{\Omega_\mathrm{H}} g_1(v_n)\partial_t w_n - \int_0^t\int_{\Omega_\mathrm{H}}\frac{c_1}{2}\partial_t w_n^2.
\end{aligned} \tag{3.38}$$

For the last three terms of the right-hand side, we proceed as follows. First, using $(2.16)_2$ and Young's inequality, we notice that

$$\begin{aligned}
\left|\int_0^t\int_{\Omega_\mathrm{H}} f_2(v_n)w_n\partial_t v_n\right| &= \left|\int_0^t\int_{\Omega_\mathrm{H}} c_4\partial_t v_n w_n + c_5 v_n\partial_t v_n w_n\right| \\
&\leq \frac{C_\mathrm{m}}{4}\|\partial_t v_n\|^2_{L^2(Q_t)} + c\|w_n\|^2_{L^2(Q_t)} + \left|\frac{c_5}{2}\int_0^t\int_{\Omega_\mathrm{H}} w_n\partial_t v_n^2\right|.
\end{aligned}$$

In addition, integration by parts in the last term with Young's inequality and Cauchy–Schwarz inequality yields

$$\begin{aligned}
\left|\frac{c_5}{2}\int_0^t\int_{\Omega_\mathrm{H}} w_n\partial_t v_n^2\right| &\leq \frac{|c_5|}{2}\left|\int_0^t\int_{\Omega_\mathrm{H}}\partial_t w_n v_n^2\right| + \frac{|c_5|}{2}\int_{\Omega_\mathrm{H}}\left|w_n(t)v_n^2(t) - w_{0,n}v_{0,n}^2\right| \\
&\leq c\|v_n\|^4_{L^4(Q_t)} + \frac{1}{4}\|\partial_t w_n\|^2_{L^2(Q_t)} + c\left(\|v_{0,n}\|^4_{L^4(\Omega_\mathrm{H})} + \|w_{0,n}\|^2_{L^2(\Omega_\mathrm{H})}\right) \\
&\quad + c\|w_n(t)\|_{L^2(\Omega_\mathrm{H})}\|v_n(t)\|^2_{L^4(\Omega_\mathrm{H})},
\end{aligned}$$

where the last term can be estimated by combining Hölder's inequality and the continuous embedding of $H^1(\Omega_\mathrm{H})$ in $L^6(\Omega_\mathrm{H})$, namely

$$\|v_n(t)\|^2_{L^4(\Omega_\mathrm{H})} \leq \|v_n(t)\|^{\frac{1}{2}}_{L^2(\Omega_\mathrm{H})}\|v_n(t)\|^{\frac{3}{2}}_{L^6(\Omega_\mathrm{H})} \leq c\|v_n(t)\|^{\frac{1}{2}}_{L^2(\Omega_\mathrm{H})}\|v_n(t)\|^{\frac{3}{2}}_{H^1(\Omega_\mathrm{H})}.$$

Finally, using $(2.16)_3$ we have,

$$\left|\int_0^t \int_{\Omega_{\rm H}} g_1(v_n)\partial_t w_n\right| \le c\big(|\Omega_{\rm H}|t + \|v_n\|^4_{L^4(Q_t)}\big) + \frac{1}{4}\|\partial_t w_n\|^2_{L^2(Q_t)},$$

and

$$\left|\int_0^t \int_{\Omega_{\rm H}} \frac{c_1}{2}\partial_t\left(w_n^2\right)\right| = \frac{|c_1|}{2}\left|\int_{\Omega_{\rm H}} w_n^2(t) - \int_{\Omega_{\rm H}} w_{0,n}^2\right| \le \frac{|c_1|}{2}\|w_n(t)\|^2_{L^2(\Omega_{\rm H})} + \frac{|c_1|}{2}\|w_{0,n}\|^2_{L^2(\Omega_{\rm H})}.$$

As a result, inserting these last estimates in (3.38), we obtain

$$\begin{aligned}
&\frac{1}{2}\|\partial_t w_n\|^2_{L^2(Q_t)} + \frac{C_{\rm m}}{4}\|\partial_t v_n\|^2_{L^2(Q_t)} + \frac{1}{n}\|\partial_t u_{{\rm i},n}\|^2_{L^2(Q_t)} + \frac{1}{n}\|\partial_t u_n\|^2_{L^2(0,t;L^2(\Omega))}\\
&\quad + \frac{\alpha_{\rm i}}{2}\|\nabla u_{{\rm i},n}\|^2_{L^2(\Omega_{\rm H})} + \frac{\alpha}{2}\|\nabla u_n\|^2_{L^2(\Omega)} \le c + \frac{1}{2C_{\rm m}}\|I_{\rm app}\|^2_{L^2(Q_t)} + c\|v_n(t)\|^2_{L^2(\Omega_{\rm H})}\\
&\quad + c\|w_n\|^2_{L^2(Q_t)} + c\|v_n\|^4_{L^4(Q_t)} + c\left(\|v_{0,n}\|^4_{L^4(\Omega_{\rm H})} + \|w_{0,n}\|^2_{L^2(\Omega_{\rm H})}\right)\\
&\quad + c\|w_n(t)\|_{L^2(\Omega_{\rm H})}\|v_n(t)\|^{\frac{1}{2}}_{L^2(\Omega_{\rm H})}\|v_n(t)\|^{\frac{3}{2}}_{H^1(\Omega_{\rm H})} + c|\Omega_{\rm H}|t + c\|w_n(t)\|^2_{L^2(\Omega_{\rm H})},
\end{aligned} \tag{3.39}$$

for all $t \in [0, T']$.

Therefore, using (3.27), the previous estimates $(3.31)_1$, $(3.32)_1$, and since $T' \le T$, inequality (3.39) reduces to

$$\begin{aligned}
\frac{1}{2}\|\partial_t w_n\|^2_{L^2(Q_t)} + \frac{C_{\rm m}}{4}\|\partial_t v_n\|^2_{L^2(Q_t)} + \frac{1}{n}\|\partial_t u_{{\rm i},n}\|^2_{L^2(Q_t)} + \frac{1}{n}\|\partial_t u_n\|^2_{L^2(0,t;L^2(\Omega))}&\\
+ \frac{\alpha_{\rm i}}{2}\|\nabla u_{{\rm i},n}\|^2_{L^2(\Omega_{\rm H})} + \frac{\alpha}{2}\|\nabla u_n\|^2_{L^2(\Omega)} \le c\left(1 + \|v_n(t)\|^{\frac{3}{2}}_{H^1(\Omega_{\rm H})}\right)&,
\end{aligned}$$

for all $t \in [0, T']$. In particular, using estimates $(3.31)_1$, we obtain

$$\frac{1}{2}\min(\alpha, \alpha_i)\|v_n(t)\|^2_{H^1(\Omega_{\rm H})} \le c\left(1 + \|v_n(t)\|^{\frac{3}{2}}_{H^1(\Omega_{\rm H})}\right),$$

so that v_n is uniformly bounded in $L^\infty(0, T'; H^1(\Omega_{\rm H}))$. Hence, we obtain the desired estimates $(3.31)_2$ and $(3.32)_2$.

Now, we consider problem $\mathbf{P2}_n$, by proving the estimate (3.33). From $(3.26)_1$ it follows that $\partial_t w_n = -g(v_n, w_n)$ and, on the other hand, according to (2.15), we have

$0 \leq h_\infty \leq 1$. Thus, from (2.13)$_2$ we have, *a.e.* in $[0, T']$,

$$\begin{aligned} \partial_t w_n &\geq -w_n\Big(\frac{1}{\tau_{\text{close}}} + \frac{\tau_{\text{close}} - \tau_{\text{open}}}{\tau_{\text{close}}\tau_{\text{open}}} h_\infty(v_n)\Big), \\ \partial_t w_n &\leq (1 - w_n)\Big(\frac{1}{\tau_{\text{close}}} + \frac{\tau_{\text{close}} - \tau_{\text{open}}}{\tau_{\text{close}}\tau_{\text{open}}} h_\infty(v_n)\Big), \end{aligned} \tag{3.40}$$

which combined with Gronwall lemma yields

$$\begin{aligned} w_n &\geq w_0 \exp\left[-\int_0^t \left(\frac{1}{\tau_{\text{close}}} + \frac{\tau_{\text{close}} - \tau_{\text{open}}}{\tau_{\text{close}}\tau_{\text{open}}} h_\infty(v_n)\right)\right], \\ w_n &\leq 1 - (1 - w_0) \exp\left[-\int_0^t \left(\frac{1}{\tau_{\text{close}}} + \frac{\tau_{\text{close}} - \tau_{\text{open}}}{\tau_{\text{close}}\tau_{\text{open}}} h_\infty(v_n)\right)\right]. \end{aligned}$$

Using (2.23), we then obtain that

$$w_{\min} \stackrel{\text{def}}{=} r \exp\left(\frac{-T}{\tau_{\text{open}}}\right) \leq w_n \leq 1, \quad a.e. \text{ in} Q_{T'}.$$

On the other hand, combining this estimate with (3.40), we get

$$\frac{-1}{\tau_{\text{open}}} \leq \partial_t w_n \leq \frac{1}{\tau_{\text{open}}}, \quad a.e. \text{ in} Q_{T'}.$$

which completes the proof of (3.33). ■

Finally, the energy estimates (3.31)$_1$ are obtained in a standard fashion by taking $h = u_{\mathrm{i},n}$ and $e = -u_n$ in (3.24)$_{1,2}$, which yields

$$\begin{aligned} \frac{1}{2}\frac{\mathrm{d}}{\mathrm{dt}}\left[C_{\mathrm{m}}\|v_n\|^2_{L^2(\Omega_{\mathrm{H}})} + \frac{1}{n}\left(\|u_{\mathrm{i},n}\|^2_{L^2(\Omega_{\mathrm{H}})} + \|u_n\|^2_{L^2(\Omega)}\right)\right] + \alpha_{\mathrm{i}}\|\nabla u_{\mathrm{i},n}\|^2_{L^2(\Omega_{\mathrm{H}})} \\ + \alpha\|\nabla u_n\|^2_{L^2(\Omega)} + \int_{\Omega_{\mathrm{H}}} I_{\mathrm{ion}}(v_n, w_n)v_n \leq \int_{\Omega_{\mathrm{H}}} I_{\mathrm{app}} v_n. \end{aligned} \tag{3.41}$$

Conversely, assumption (2.17) and estimate (3.33) lead to

$$I_{\mathrm{ion}}(v, w)v \geq \frac{a}{\tau_{\mathrm{in}}} w_{\min}|v|^4 - \left(\frac{b}{\tau_{\mathrm{in}}} + \frac{1}{\tau_{\mathrm{out}}}\right)|v|^2,$$

so that, from (3.41), we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{dt}}\left[C_{\mathrm{m}}\|v_n\|^2_{L^2(\Omega_{\mathrm{H}})}+\frac{1}{n}\left(\|u_{\mathrm{i},n}\|^2_{L^2(\Omega_{\mathrm{H}})}+\|u_n\|^2_{L^2(\Omega)}\right)\right]$$
$$+\alpha_{\mathrm{i}}\|\nabla u_{\mathrm{i},n}\|^2_{L^2(\Omega_{\mathrm{H}})}+\alpha\|\nabla u_n\|^2_{L^2(\Omega)}+\frac{a}{\tau_{in}}w_{\min}\|v_n\|^4_{L^4(\Omega_{\mathrm{H}})}$$
$$\leq\left(\frac{b}{\tau_{\mathrm{in}}}+\frac{1}{\tau_{\mathrm{out}}}+\frac{1}{2}\right)\|v_n\|^2_{L^2(\Omega_{\mathrm{H}})}+\frac{1}{2}\|I_{\mathrm{app}}\|^2_{L^2(\Omega_{\mathrm{H}})}.$$

We then obtain the energy estimate (3.31))$_1$ by applying Gronwall lemma.

For the estimate on the time derivatives, we take $h=\partial_t u_{\mathrm{i},n}$ and $e=\partial_t u_n$ in (3.24) and we integrate over $(0,t)$, with $t\in[0,T']$. Using Cauchy–Schwarz and Young's inequalities, we obtain

$$\frac{C_{\mathrm{m}}}{4}\|\partial_t v_n\|^2_{L^2(Q_t)}+\frac{1}{n}\left(\|\partial_t u_{\mathrm{i},n}\|^2_{L^2(Q_t)}+\|\partial_t u_n\|^2_{L^2(0,t;L^2(\Omega))}\right)+\frac{\alpha_{\mathrm{i}}}{2}\|\nabla u_{\mathrm{i},n}\|_{L^2(\Omega_{\mathrm{H}})}$$
$$+\frac{\alpha}{2}\|\nabla u_n\|_{L^2(\Omega)}\leq c\left(\|\nabla u_{\mathrm{i},0,n}\|^2_{L^2(\Omega_{\mathrm{H}})}+\|\nabla u_{0,n}\|^2_{L^2(\Omega)}\right)+\frac{1}{2C_{\mathrm{m}}}\|I_{\mathrm{app}}\|^2_{L^2(Q_t)}$$
$$+\frac{1}{\tau^2_{\mathrm{out}}C_{\mathrm{m}}}\|v_n\|^2_{L^2(Q_t)}-\frac{1}{\tau_{\mathrm{in}}}\int_0^t\int_{\Omega_{\mathrm{H}}}w_n f_1(v_n)\partial_t v_n. \tag{3.42}$$

On the other hand, using the same notation in (3.35) and the fact that f_1 satisfies (2.16)$_1$, the same argument is used to obtain the inequality (3.37). Integrating by parts the last term of (3.42), we have

$$-\int_0^t\int_{\Omega_{\mathrm{H}}}w_n f_1(v_n)\partial_t v_n=-\int_0^t\int_{\Omega_{\mathrm{H}}}w_n\partial_t H(v_n)$$
$$=-\int_{\Omega_{\mathrm{H}}}w_n H(v_n)+\int_{\Omega_{\mathrm{H}}}w_0 H(v_{0,n})+\int_0^t\int_{\Omega_{\mathrm{H}}}\partial_t w_n H(v_n)$$
$$\leq c\|w_n(t)\|_{L^\infty(\Omega_{\mathrm{H}})}\|v_n(t)\|^2_{L^2(\Omega_{\mathrm{H}})}$$
$$+c\|w_0\|_{L^\infty(\Omega_{\mathrm{H}})}\left(1+\|v_{0,n}\|^4_{L^4(\Omega_{\mathrm{H}})}\right)$$
$$+c\|\partial_t w_n\|_{L^\infty(Q_t)}\left(1+\|v_n\|^4_{L^4(Q_t)}\right).$$

Therefore, inserting this estimate in (3.42), using (3.27) and the previous estimates (3.31)$_1$ and (3.33), we obtain (3.31)$_2$, which completes the proof of Lemma 3.3. ■

3.4 Weak solution of the bidomain-torso problem

First of all, we notice that energy estimates allow to extend the existence time of our discrete solution $(u_{i,n}, u_n, w_n)$. Indeed, according to Lemma 3.3, the solution satisfies, for all $t \in [0, T']$ where T' is the existence time,

$$\|u_{i,n}(t)\|_{H^1(\Omega_{\rm H})} + \|u_n(t)\|_{H^1(\Omega)} + \|w_n(t)\|_{L^2(\Omega_{\rm H})} \leq C_1.$$

Applying iteratively Lemma 3.1, we thus obtain the existence of solution up to an arbitrary time T.

We want now to pass to the limit when n goes to infinity. We first consider problem **P1**. Let us multiply (3.24) by a function $\alpha \in \mathcal{D}(0, T)$ and integrate between 0 and T. For all $k \leq n$, we have

$$\begin{aligned} &C_{\rm m} \int_0^T \int_{\Omega_{\rm H}} \alpha\, \partial_t v_n h_k + \frac{1}{n} \int_0^T \int_{\Omega_{\rm H}} \alpha\, \partial_t u_{{\rm i},n} h_k + \int_0^T \int_{\Omega_{\rm H}} \alpha\, \boldsymbol{\sigma}_{\rm i} \nabla u_{{\rm i},n} \cdot \nabla h_k \\ &+ \int_0^T \int_{\Omega_{\rm H}} \alpha\, I_{\rm ion}(v_n, w_n) h_k = \int_0^T \int_{\Omega_{\rm H}} \alpha\, I_{\rm app} h_k, \end{aligned} \tag{3.43}$$

$$\begin{aligned} &C_{\rm m} \int_0^T \int_{\Omega_{\rm H}} \alpha\, \partial_t v_n e_k - \frac{1}{n} \int_0^T \int_{\Omega} \alpha\, \partial_t u_n e_k - \int_0^T \int_{\Omega} \alpha\, \boldsymbol{\sigma} \nabla u_n \cdot \nabla e_k \\ &+ \int_0^T \int_{\Omega_{\rm H}} \alpha\, I_{\rm ion}(v_n, w_n) e_k = \int_0^T \int_{\Omega_{\rm H}} \alpha\, I_{\rm app} e_k \end{aligned} \tag{3.44}$$

$$\int_0^T \int_{\Omega_{\rm H}} \alpha\, \partial_t w_n h_k + \int_0^T \int_{\Omega_{\rm H}} \alpha\, g(v_n, w_n) h_k = 0. \tag{3.45}$$

From Lemma 3.3, it follows that there exists four functions $u \in L^\infty(0, T; V)$, $v_{\rm m} \in L^\infty(0, T; H^1(\Omega_{\rm H})) \cap L^4(Q_T) \cap H^1(0, T; L^2(\Omega_{\rm H}))$, $u_{\rm i} \in L^\infty(0, T; H^1(\Omega_{\rm H}))$ and $w \in H^1(0, T; L^2(\Omega_{\rm H}))$

such that, up to extracted subsequences, we have:

$$\begin{cases} u_n \to u \text{ in } L^\infty(0,T;V) \text{ weak } *, \\ v_n \to v_{\mathrm{m}} \text{ in } L^\infty(0,T;H^1(\Omega_{\mathrm{H}})) \text{ weak } *, \\ v_n \to v_{\mathrm{m}} \text{ weakly in } L^4(Q_T), \\ v_n \to v_{\mathrm{m}} \text{ weakly in } H^1(0,T;L^2(\Omega_{\mathrm{H}})), \\ u_{\mathrm{i},n} \to u_{\mathrm{i}} \text{ in } L^\infty(0,T;H^1(\Omega_{\mathrm{H}})) \text{ weak } *, \\ w_n \to w \text{ weakly in } H^1(0,T;L^2(\Omega_{\mathrm{H}})). \end{cases} \tag{3.46}$$

Moreover, according to Lemma 3.3, we also notice that $\frac{1}{\sqrt{n}} u_{\mathrm{i},n}$ and $\frac{1}{\sqrt{n}} u_n$ are bounded in $L^\infty(0,T;L^2(\Omega_{\mathrm{H}}))$ and $L^\infty(0,T;L^2(\Omega))$, respectively. Thus, for all $k \in \mathbb{N}^*$ and $\alpha \in \mathcal{D}(0,T)$, we have

$$\lim_{n\to+\infty} \frac{1}{n} \int_0^T \int_{\Omega_{\mathrm{H}}} \alpha\, \partial_t u_{\mathrm{i},n} h_k = 0, \quad \lim_{n\to+\infty} \frac{1}{n} \int_0^T \int_{\Omega} \alpha\, \partial_t u_n e_k = 0.$$

Let us consider now the nonlinear terms in (3.43)–(3.45). Since $\{v_n\}$ is bounded in $L^2(0,T;H^1(\Omega_{\mathrm{H}})) \cap H^1(0,T;L^2(\Omega_{\mathrm{H}}))$, we have that $\{v_n\}$ is bounded in $H^1(Q_T)$. Hence, thanks to the compact embedding of $H^1(Q_T)$ in $L^3(Q_T)$, the sequence $\{v_n\}$ strongly converges to v_{m} in $L^3(Q_T)$. In addition, using the Lebesgue's dominated convergence theorem, we deduce that there exists a positive function $\mathcal{V} \in L^1(Q_T)$ such that, up to extraction, $v_n^3 \le \mathcal{V}$ and that $v_n \to v_{\mathrm{m}}$ *a.e.* in Q_T. Thus, from $(2.16)_1$ and using once again the Lebesgue's dominated convergence theorem, it follows that $\{f_1(v_n)\}$ strongly converges to $f_1(v_{\mathrm{m}})$ in $L^1(Q_T)$. As a result,

$$\lim_{n\to+\infty} \int_0^T \int_{\Omega_{\mathrm{H}}} \alpha\, f_1(v_n) h_k = \int_0^T \int_{\Omega_{\mathrm{H}}} \alpha\, f_1(v_{\mathrm{m}}) h_k.$$

On the other hand, since $\{w_n\}$ is bounded in $L^2(Q_T)$ and $\{v_n\}$ strongly converges to v_{m} in $L^2(Q_T)$, we have

$$\lim_{n\to+\infty} \int_0^T \int_{\Omega_{\mathrm{H}}} \alpha\, f_2(v_n) w_n h_k = \int_0^T \int_{\Omega_{\mathrm{H}}} \alpha\, f_2(v_{\mathrm{m}}) w h_k.$$

Thus, in summary,

$$\lim_{n\to+\infty} \int_0^T \int_{\Omega_{\mathrm{H}}} \alpha\, I_{\mathrm{ion}}(v_n, w_n) h_k = \int_0^T \int_{\Omega_{\mathrm{H}}} \alpha\, I_{\mathrm{ion}}(v_{\mathrm{m}}, w) h_k.$$

Similar arguments allow us to prove that

$$\lim_{n\to+\infty}\int_0^T\int_{\Omega_{\mathrm{H}}}\alpha\, g(v_n)h_k=\int_0^T\int_{\Omega_{\mathrm{H}}}\alpha\, g(v_{\mathrm{m}})h_k.$$

We can then pass to the limit in n in (3.43)–(3.45), yielding

$$\begin{aligned} C_{\mathrm{m}}\int_0^T\int_{\Omega_{\mathrm{H}}}\alpha\,\partial_t v_{\mathrm{m}}h_k+\int_0^T\int_{\Omega_H}\alpha\,\boldsymbol{\sigma}_{\mathrm{i}}\nabla u_{\mathrm{i}}\cdot\nabla h_k \\ +\int_0^T\int_{\Omega_H}\alpha\, I_{\mathrm{ion}}(v_{\mathrm{m}},w)h_k=\int_0^T\int_{\Omega_H}\alpha\, I_{\mathrm{app}}h_k, \end{aligned} \tag{3.47}$$

$$\begin{aligned} C_{\mathrm{m}}\int_0^T\int_{\Omega_{\mathrm{H}}}\alpha\,\partial_t v_{\mathrm{m}}e_k-\int_0^T\int_{\Omega}\alpha\,\boldsymbol{\sigma}\nabla u\cdot\nabla e_k \\ +\int_0^T\int_{\Omega_H}\alpha\, I_{\mathrm{ion}}(v_{\mathrm{m}},w)e_k=\int_0^T\int_{\Omega_H}\alpha\, I_{\mathrm{app}}e_k, \end{aligned} \tag{3.48}$$

$$\int_0^T\int_{\Omega_{\mathrm{H}}}\alpha\,\partial_t w h_k+\alpha\, g(v_{\mathrm{m}},w)h_k=0, \tag{3.49}$$

for all $k\in\mathbb{N}^*$ and $\alpha\in\mathcal{D}(0,T)$. We obtain (2.20)–(2.22) from the density properties of the spaces spanned by $\{h_k\}_{k\in\mathbb{N}^*}$ and $\{e_k\}_{k\in\mathbb{N}^*}$.

Finally, it only remains to be proved that v_{m} and w satisfy the initial conditions (1.5). Since (v_n) weakly converges to v_{m} in $H^1(0,T;L^2(\Omega_{\mathrm{H}}))$, (v_n) strongly converges to v_{m} in $C(0,T;H^{-1}(\Omega_{\mathrm{H}}))$ for instance. This allows to assert that $v_{\mathrm{m}}(0)=v_0$ in Ω_{H} since, by construction, $v_n(0)\to v_0$ in $L^2(\Omega_{\mathrm{H}})$. The same argument holds for w.

For problem **P2**, the arguments of passing to the limit can be adapted without major modifications. For the nonlinear terms, we can (as previously) prove that $\{v_n\}$ strongly converges to v_{m} in $L^3(Q_T)$. Thus $f_1(v_n)$ strongly converges to $f_1(v_{\mathrm{m}})$ in $L^1(Q_T)$. Since

$$w_n\to w \text{ in } L^\infty(Q_T) \text{ weak} \star,$$

this allows us to prove that

$$\lim_{n\to+\infty}\int_0^T\int_{\Omega_{\mathrm{H}}}\alpha\, I_{\mathrm{ion}}(v_n,w_n)h_k=\int_0^T\int_{\Omega_{\mathrm{H}}}\alpha\, I_{\mathrm{ion}}(v_{\mathrm{m}},w)h_k.$$

Moreover, since $h_\infty(v_n) \to h_\infty(v_{\rm m})$ a.e. in Q_T and $\{h_\infty(v_n)\}$ is bounded in $L^\infty(Q_T)$, $\{h_\infty(v_n)\}$ strongly converges in $L^2(Q_T)$ to $h_\infty(v_{\rm m})$. Thus we can also pass to the limit in Equation (3.26). This allows us to obtain a weak solution of **P2** as defined by Definition 2.1.

3.5 Uniqueness of the weak solution

In this paragraph, we prove the uniqueness of weak solution for problem **P1**, under the additional assumption **A3**. This is a direct consequence of the following *comparison Lemma*.

Lemma 3.4. Assume that assumption **A3** holds and that

$$(v_{\rm m,1}, u_{\rm i,1}, u_1, w_1), \quad (v_{\rm m,2}, u_{\rm i,2}, u_2, w_2),$$

are two weak solutions of problem **P1** corresponding, respectively, to the initial data $(v_{1,0}, w_{1,0})$ and $(v_{2,0}, w_{2,0})$, and right-hand sides $I_{\rm app,1}$ and $I_{\rm app,2}$. For all $t \in (0, T)$, there holds

$$\begin{aligned}
&\|v_1(t) - v_2(t)\|^2_{L^2(\Omega_{\rm H})} + \|w_1(t) - w_2(t)\|^2_{L^2(\Omega_{\rm H})} \\
&\qquad \le \exp\left(K_1 t\right) K_2 \Big(\|v_{1,0} - v_{2,0}\|^2_{L^2(\Omega_{\rm H})} + |w_{1,0} - w_{2,0}|^2_{L^2(\Omega_{\rm H})} + \|I_{\rm app,1} - I_{\rm app,2}\|^2_{L^2(Q_t)} \Big),
\end{aligned}$$

with $K_1, K_2 > 0$ positive constants only depending on $C_{\rm m}$, μ_0, and $C_{\rm ion}$. □

Proof. The proof follows the argument provided in [5] for the isolated bidomain equations. According to Definition 2.1, we have, for all $\phi_{\rm i} \in L^2(0, T; H^1(\Omega_{\rm H}))$, $\psi \in L^2(0, T; V)$ and $\theta \in L^2(0, T; L^2(\Omega_{\rm H}))$,

$$\begin{aligned}
&C_{\rm m} \int_0^t \int_{\Omega_{\rm H}} \partial_t (v_1 - v_2)\phi_{\rm i} + \int_0^t \int_{\Omega_{\rm H}} \boldsymbol{\sigma}_{\rm i} (\nabla u_{\rm i,1} - \nabla u_{\rm i,2}) \cdot \nabla \phi_{\rm i} \\
&\qquad + \int_0^t \int_{\Omega_{\rm H}} (I_{\rm ion}(v_1, w_1) - I_{\rm ion}(v_2, w_2))\phi_{\rm i} = \int_0^t \int_{\Omega_{\rm H}} (I_{\rm app,1} - I_{\rm app,2})\phi_{\rm i}, \\
&C_{\rm m} \int_0^t \int_{\Omega_{\rm H}} \partial_t (v_1 - v_2)\psi - \int_0^t \int_{\Omega} \boldsymbol{\sigma} (\nabla u_1 - \nabla u_2) \cdot \nabla \psi \\
&\qquad + \int_0^t \int_{\Omega_{\rm H}} (I_{\rm ion}(v_1, w_1) - I_{\rm ion}(v_2, w_2))\psi = \int_0^t \int_{\Omega_{\rm H}} (I_{\rm app,1} - I_{\rm app,2})\psi, \\
&\int_0^t \int_{\Omega_{\rm H}} \partial_t (w_1 - w_2)\theta + \int_0^t \int_{\Omega_{\rm H}} (g(v_1, w_1) - g(v_2, w_2))\theta = 0.
\end{aligned}$$

For $\mu > 0$, we take in this expression $\phi_{\mathrm{i}} = \mu(u_{\mathrm{i},1} - u_{\mathrm{i},2})$, $\psi = -\mu(u_1 - u_2)$ and $\theta = w_1 - w_2$. Thus, adding the resulting equalities, we have

$$\begin{aligned}\frac{\mu C_{\mathrm{m}}}{2}\|v_1(t) - v_2(t)\|^2_{L^2(\Omega_{\mathrm{H}})} + \frac{1}{2}\|w_1(t) - w_2(t)\|^2_{L^2(\Omega_{\mathrm{H}})}&\\ + \mu\left(\alpha_{\mathrm{i}}\|\nabla(u_{\mathrm{i},1} - u_{\mathrm{i},2})\|^2_{L^2(Q_t)} + \alpha\|\nabla(u_1 - u_2)\|^2_{L^2(\Omega\times(0,t))}\right)&\\ + \mu\int_0^t\int_{\Omega_{\mathrm{H}}}(I_{\mathrm{ion}}(v_1, w_1) - I_{\mathrm{ion}}(v_2, w_2))(v_1 - v_2)&\\ + \int_0^t\int_{\Omega_{\mathrm{H}}}(g(v_1, w_1) - g(v_2, w_2))(w_1 - w_2)&\\ \leq \frac{\mu C_{\mathrm{m}}}{2}\|v_{1,0} - v_{2,0}\|^2_{L^2(\Omega_{\mathrm{H}})} + \frac{1}{2}\|w_{1,0} - w_{2,0}\|^2_{L^2(\Omega_{\mathrm{H}})}&\\ + \frac{\mu^2}{2}\|I_{\mathrm{app},1} - I_{\mathrm{app},2}\|^2_{L^2(Q_t)} + \frac{1}{2}\|v_1 - v_2\|^2_{L^2(Q_t)}.&\end{aligned} \tag{3.50}$$

Let $\mu_0 > 0$ be the parameter provided by assumption **A3**. We define

$$\begin{aligned}\Phi(v_1, w_1, v_2, w_2) \stackrel{\mathrm{def}}{=} \int_{\Omega_{\mathrm{H}}}\mu_0(I_{\mathrm{ion}}(v_1, w_1) - I_{\mathrm{ion}}(v_2, w_2))(v_1 - v_2)&\\ + \int_{\Omega_{\mathrm{H}}}(g(v_1, w_1) - g(v_2, w_2))(w_1 - w_2),&\end{aligned} \tag{3.51}$$

Denoting $z \stackrel{\mathrm{def}}{=} (v, w)$ and using **A3**, we have

$$\Phi(v_1, w_1, v_2, w_2) = \Phi(z_1, z_2) = \int_{\Omega_{\mathrm{H}}}\big(F_{\mu_0}(z_1) - F_{\mu_0}(z_2)\big)\cdot(z_1 - z_2).$$

Since F_{μ_0} is continuously differentiable, a Taylor expansion with integral remainder yields

$$F_{\mu_0}(z_1) - F_{\mu_0}(z_2) = \int_0^1 \nabla F_{\mu_0}\big(\xi z_1 + (1-\xi)z_2\big)\cdot(z_1 - z_2)\,\mathrm{d}\xi, \quad \forall\, z_1, z_2 \in \mathbb{R}^2.$$

Inserting this expression in (3.51) and using the assumed spectral bound (2.18), there follows

$$\begin{aligned}\Phi(z_1, z_2) &= \int_0^1\int_{\Omega_{\mathrm{H}}}(z_1 - z_2)\cdot\nabla F_{\mu_0}(\xi z_1 + (1-\xi)z_2)\cdot(z_1 - z_2)\,\mathrm{d}\xi\\ &\geq C_{\mathrm{ion}}\int_0^1\|z_1 - z_2\|^2_{L^2(\Omega_{\mathrm{H}})}\,\mathrm{d}\xi\\ &= C_{\mathrm{ion}}\big(\|v_1 - v_2\|^2_{L^2(\Omega_{\mathrm{H}})} + \|w_1 - w_2\|^2_{L^2(\Omega_{\mathrm{H}})}\big).\end{aligned}$$

Therefore, from (3.50) with $\mu = \mu_0$, we have

$$\begin{aligned}
&\frac{\mu_0 C_{\mathrm{m}}}{2}\|v_1(t) - v_2(t)\|^2_{L^2(\Omega_{\mathrm{H}})} + \frac{1}{2}\|w_1(t) - w_2(t)\|^2_{L^2(\Omega_{\mathrm{H}})} \\
&\leq \frac{\mu C_{\mathrm{m}}}{2}\|v_{1,0} - v_{2,0}\|^2_{L^2(\Omega_{\mathrm{H}})} + \frac{1}{2}\|w_{1,0} - w_{2,0}\|^2_{L^2(\Omega_{\mathrm{H}})} + \frac{\mu^2}{2}\|I_{\mathrm{app},1} - I_{\mathrm{app},2}\|^2_{L^2(Q_t)} \\
&\qquad + \left|\frac{1}{2} - C_{\mathrm{ion}}\right| \|v_1 - v_2\|^2_{L^2(Q_t)} + |C_{\mathrm{ion}}|\|w_1 - w_2\|^2_{L^2(Q_t)}.
\end{aligned} \tag{3.52}$$

We conclude the proof using Gronwall Lemma. ■

Acknowledgments

We would like to thank the anonymous referees for their careful reading and their constructive comments.

References

[1] Aliev, R., and A. Panfilov. "A simple two-variable model of cardiac excitation." *Chaos, Solitons and Fractals* 7 (1996): 293–301.

[2] Bendahmane, M., and K. Karlsen. "Analysis of a class of degenerate reaction-diffusion systems and the bidomain model of cardiac tissue." *Network and Heterogenous Media* 1, no. 1 (2006): 185–218.

[3] Boulakia, M., M. A. Fernández, J.-F. Gerbeau, and N. Zemzemi. "Mathematical modelling of electrocardiograms: a numerical study." (forthcoming).

[4] Boulakia, M., M. A. Fernández, J.-F. Gerbeau, and N. Zemzemi. "Towards the numerical simulation of electrocardiograms." In *Functional Imaging and Modeling of the Heart*, edited by F. B. Sachse and G. Seemann, 240–49. Lecture Notes in Computer Science 4466. Berlin: Springer-Verlag, 2007.

[5] Bourgault, Y., Y. Coudière, and C. Pierre. "Well-posedness of a parabolic problem based on a bidomain model for electrophysiological wave propagation." *Nonlinear Analalysis: Real World Applications* (forthcoming).

[6] Cartan, H. *Calcul différentiel*. Paris: Hermann, 1967.

[7] Colli, P. Franzone and G. Savaré. "Degenerate evolution systems modeling the cardiac electric field at micro- and macroscopic level." In *Evolution Equations, Semigroups and Functional Analysis (Milano, 2000)*. Special issue, *Progress in Nonlinear Differential Equations and Their Applications* 50 (2002): 49–78. Birkhäuser, 2002.

[8] Dautray, R., and J.-L. Lions. *Analyse mathématique et calcul numérique pour les sciences et les techniques, Tome 3*. Paris: Masson, 1985.

[9] Djabella, K., M. Sorine, and M. Landau. "A Two-Variable Model of Cardiac Action Potential with Controlled Pacemaker Activity and Ionic Current Interpretation." 64th IEEE conference, New Orleans, Louisiana USA, 2007.

[10] Fitzhugh, R. “Impulses and physiological states in theoretical models of nerve membrane.” *Biophysical Journal* 1 (1961): 445–65.

[11] Gulrajani, R. M. “Computer Heart Models and the Simulation of the Electrocardiogram: Newer Strategies,” 17–21. 25th Annual International Conference of the IEEE EMBS, 2003.

[12] Krassowska, W., and J. C. Neu. “Homogenization of syncitial tissues.” *CRC Critical Reviews in Biomedical Engineering* 21, no. 2 (1993): 137–99.

[13] Krassowska, W., and J. C. Neu. “Effective boundary conditions for syncitial tissues.” *IEEE Transactions in Biomedical Engineering* 41, no. 2 (1994): 137–99.

[14] Luo, C. H., and Y. Rudy. “A model of the ventricular cardiac action potential. Depolarization, repolarization, and their interaction.” *Circulation Research* 68 (1991): 1501–26.

[15] Luo, C. H., and Y. Rudy. “A dynamic model of the cardiac ventricular action potential. Simulations of ionic currents and concentration changes.” *Circulation Research* 74 (1994): 1071–97.

[16] Mitchell, C. C., and D. G. Schaeffer. “A two-current model for the dynamics of cardiac membrane.” *Bulletin of Mathematical Biology* 65 (2003): 767–93.

[17] Nagumo, J. S., S. Arimoto, and S. Yoshizawa. “An active pulse transmission line stimulating nerve axon.” *Proceedings of IRE* 50 (1962): 2061–71.

[18] Pennacchio, M., G. Savaré, and P. Colli Franzone. “Multiscale modeling for the bioelectric activity of the heart.” *SIAM Journal of Mathematical Analysis* 37, no. 4 (2006): 1333–70.

[19] Pierre, C. Modélisation et simulation de l’activité électrique du cœur dans le thorax, analyse numérique et méthodes de volumes finis. PhD thesis, Laboratoire J. Leray, Université de Nantes, 2005.

[20] Pullan, A. J., M. L. Buist, and L. K. Cheng. *Mathematically Modelling the Electrical Activity of the Heart: From Cell to Body Surface and Back Again*. Hackensack, NJ: World Scientific Publishing, 2005.

[21] Renardy, M., and R. C. Rogers. *An Introduction to Partial Differential Equations*, 2nd ed. Texts in Applied Mathematics 13. New York: Springer, 2004.

[22] Roger, J. M., and A. D. McCulloch. “A collocation-Galerkin finite element model of cardiac action potential propagation.” *IEEE Transactions in Biomedical Engineering* 41, no. 8 (1994): 743–57.

[23] Sachse, F. B. *Computational Cardiology: Modeling of Anatomy, Electro-physiology and Mechanics*. Berlin: Springer-Verlag, 2004.

[24] Sundnes, J., G. T. Lines, X. Cai, B.F. Nielsen, K.-A. Mardal, and A. Tveito. *Computing the Electrical Activity in the Heart*. Berlin: Springer-Verlag, 2006.

[25] Tung, L. “A Bi-Domain Model for Describing Ischemic Myocardial D–C Potentials.” PhD thesis, MIT, 1978.

[26] Veneroni, M. “Reaction-diffusion systems for the macroscopic bidomain model of the cardiac electric field.” *Nonlinear Analysis: Real World Applications* (forthcoming).